Waves, history and applications[1]

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FAU - AvH

April 1, 2020

[1] Out of a former course delivered in collaboration with Aurora Marica (Polytechnique University of Bucharest)

Witham G.B., Linear and nonlinear waves, John Wiley & Sons, 1974.

"There appears to be no single precise definition of what exactly constitutes a wave. Various restrictive definitions can be given, but to cover the whole range of wave phenomena it seems preferable to be guided by the intuitive view that a wave is any recognizable signal that is transferred from one part of the medium to another with a recognizable velocity of propagation. The signal may be any feature of the disturbance, such as a maximum or an abrupt change in some quantity, provided that it can be clearly recognized and its location at any time can be determined. The signal may distort, change its magnitude and change its velocity provided it is still recognizable. This may seem a little vague, but it turns out to be perfectly adequate and any attempt to be more precise appears to be too restrictive; different features are important in different types of waves."

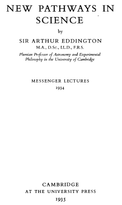
Bishop R.E.D., Vibration, Cambridge University Press, 1979.

"After all, our hearts beat, our lungs oscillate, we shiver when we are cold, we sometimes snore, we can hear and speak because our eardrums and our larynges vibrate. The light waves which permit us to see entail vibration. We move by oscillating our legs. We cannot even say *vibration* properly without the tip of the tongue oscillating. And the matter does not end there - far from it. Even the atoms of which we are constituted vibrate.

...if we are prepared to stretch the definition of vibration a little, it quickly becomes apparent that many of the events of everyday life have an extraordinarily cyclic quality. It is a curiously shaky world we live in.

It is no exaggeration to say that it is unlikely that there is any branch of science in which vibration does not play an important role."

Eddington A., New pathways in science, Cambridge University Press, 1935.


NEW PATHWAYS IN SCIENCE

by

SIR ARTHUR EDDINGTON

M.A., D.Sc., LL.D., F.R.S.

Plumian Professor of Astronomy and Experimental Philosophy in the University of Cambridge

MESSENGER LECTURES

1934

CAMBRIDGE

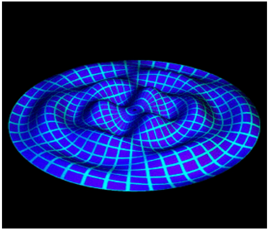
AT THE UNIVERSITY PRESS

1935

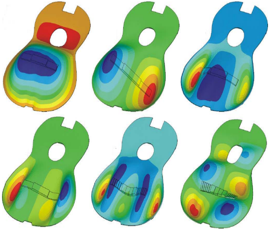
"Is the ocean composed of water or of waves or of both? Some of my fellow passengers on the Atlantic were emphatically of the opinion that it is composed of waves; but I think the ordinary unprejudiced answer would be that it is composed of water. At least if we declare our belief that the nature of the ocean is aqueous, it is not likely that anyone will challenge us and assert that on the contrary its nature is undulatory, or that it is a dualism part aqueous and part undulatory."

Ubiquity of waves: acoustic and light waves

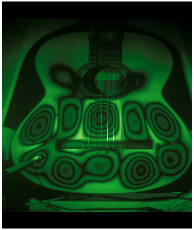
"The Book of Nature is written in the language of mathematics." (Galileo Galilei, 1564-1642)



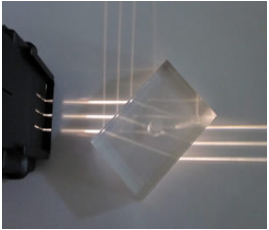
1 Eigenmode of a drum



2 Harmonics of a violin



3 Interferometry of a violin



4 Reflection and refraction of light rays in the lab

5 Reflection and refraction of light rays in a glass

"Profound study of nature is the most fertile source of mathematical discoveries." (Joseph Fourier, 1768-1830)

6 Atmospheric shock wave during a volcano eruption, Saychev Peak, Matua Island, 2009

7 Aircraft sonic boom

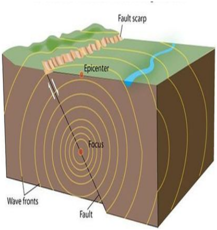
8 Impact of the shock wave on the water surface Iowa battleship

9 Shock wave when shooting

10 Small waves in shallow waters

11 Sea waves and surfing

"Mathematical analysis is as extensive as nature itself." (Joseph Fourier, 1768-1830)

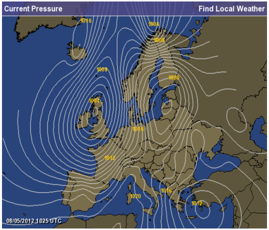


12 Seismic waves

13 Distorted railway line after Canterbury earthquake, New Zealand, 2010

14 Cracks in the street after earthquake

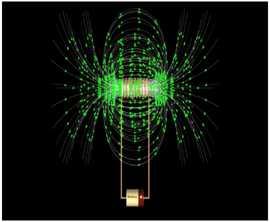
15 Mechanical waves in the water



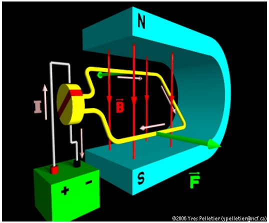
16 Atmospheric pressure waves

Ubiquity of waves: electromagnetic waves, radio, radar, wireless...

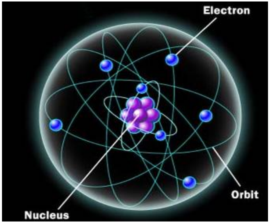
Mathematics is the alphabet with which the universe was created. (Galileo Galilei, 1564-1642)



17 Electromagnetic field

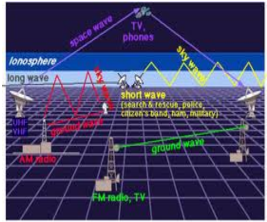


18 Propagation of electromagnetic waves in a motor



19 Quantum waves

20 Radar of a plane



21 Radio waves

22 Wireless communication is due to radio waves

Euclid (300BC) was a Greek mathematician, often referred to as the father of geometry. Although many of the results in Elements were originated by earlier mathematicians, one of Euclid's accomplishments was to present them in a coherent framework, including a system of rigorous mathematical proofs that remains the basis of mathematics 23 centuries later.

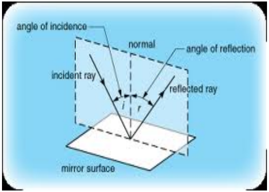
The book Catoptrics attributed to Euclid concerns the mathematical theory of mirrors. He describe the laws of reflection of light in plane and spherical concave mirrors:

- The incident ray, the reflected ray and the normal to the reflection surface at the point of the incidence lie in the same plane.
- The angle which the incident ray makes with the normal is equal to the angle which the reflected ray makes to the same normal.
- The reflected ray and the incident ray are on the opposite sides of the normal.

The first practical catoptric telescope (or Newtonian reflector) was built by Newton as a solution to the problem of chromatic aberration exhibited in telescopes using lenses as objectives (dioptric telescopes).

Optics is the earliest surviving Greek treatise on perspective:

Vision is caused by rays emanating from the eye. The eye sees objects that are within its visual cone. The visual cone is made up of straight lines.



History of waves: Heron

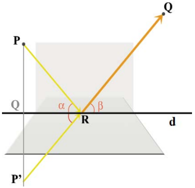
Heron de Alexandria (10-80AD) formulated the principle of the shortest path of light:

> A ray of light propagates within the same medium following the shortest possible path.

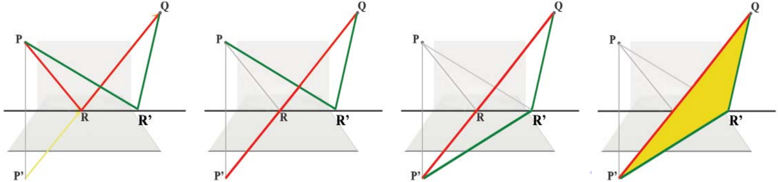
Heron's idea to find the shortest path needed by a ray to travel from P to Q by reflecting on d

▶ **CONSTRUCTION** of the incidence point R:

- Find the symmetric P' of P with respect to d
- Join P' to Q and set $R := P'Q \cap d$
- R is the incidence point between the ray of light and d.
- Consequence: the incidence angle $\alpha =$ the reflection angle β.

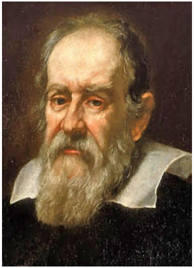


▶ **OPTIMALITY of** R**:** $PR + RQ$ is the shortest of all possible paths $PR' + R'Q$.
By symmetry, $PR = P'R$ and $PR' = P'R'$. In the triangle $R'P'Q$, $P'Q < P'R' + R'Q$:



History of waves: Galileo

Galileo Galilei (1564-1642) was an Italian physicist, mathematician, astronomer and philosopher who played a major role in the Scientific Revolution. He is considered the father of modern observational astronomy and of modern science (by Einstein and Hawking).



In 1581, when studying medicine, he noticed a swinging chandelier, which air currents shifted about to swing in larger and smaller arcs. It seemed, by comparison with his heartbeat, that the chandelier took the same amount of time to swing back and forth, no matter how far it was swinging. When he returned home, he set up two pendulums of equal length and swung one with a large sweep and the other with a small sweep and found that they kept time together.

As teacher of geometry, mechanics and astronomy at the University of Padua between 1592-1610, he made significant discoveries in pure fundamental science (kinematics and astronomy) and in applied science (an improvement of the telescope). He is the first scientist looking the sky using a telescope. He found that Jupiter has satellites.

In 1615, cardinal Bellarmine asked a physical proof of the heliocentric Copernican theory against the geocentric one of Ptolemy. Galileo considered his theory of the tides to provide the required proof of the motion of the Earth around Sun and wrote Dialogue on the Two Chief World Systems. For Galileo, the tides were caused the Earth's rotation on its axis and revolution around the Sun. Eppur si muove! Galileo has problems with the church, is taken in front of the Pope, forced to say that Earth does not move and died in prison.

The unreasonable effectiveness of mathematics in the natural sciences (Wigner, 1959). Galileo - Il Saggiatore (1623):
"The universe...is written in mathematical language and the letters are triangles, circles and other geometrical figures."

Johannes Kepler (1571-1630) was a German mathematician, astronomer and astrologer.

Astronomia nova is a book published in 1609, containing results on the motion of Mars. One of the greatest books on astronomy, Astronomia nova provided strong arguments for heliocentrism and contributed valuable insight into the movement of the planets, including the first mention of their elliptical path.

Epitome of Copernican Astronomy, published in three parts between 1618 and 1621, contains the three laws of planetary motion:

- The orbit of every planet is an ellipse with the Sun at one of the two foci.
- A line joining a planet and the Sun sweeps out equal areas during equal intervals of time.
- The square of the orbital period of a planet is directly proportional to the cube of the semi-major axis of its orbit.

Christian Huygens (1629-1695) was a Dutch mathematician, astronomer, physicist and horologist. His work includes studies by telescope on the nature of the rings of Saturn, coupled oscillations and centrifugal force.

In 1690, he published Treatise on light proposing the Huygens-Fresnel principle stating that every point to which a luminous disturbance reaches becomes a source of a spherical wave. He was able to derive the laws of reflection and refraction, but could not explain the diffraction effects.

He designed more accurate clocks than the ones available at that time, suitable for sea navigation. His invention of the pendulum clock, patented in 1657, was a breakthrough in timekeeping.

In 1673, he published his mathematical analysis of pendulums, Horologium Oscillatorium sive de motu pendulorum. He analysed the tautochrone problem by finding the cycloid shape of the curve down which a mass will slide under the influence of gravity in the same amount of time regardless of its starting point.

Huygens was the first to derive the formula for the period of an ideal mathematical pendulum,

$$T = 2\pi\sqrt{l/g},$$

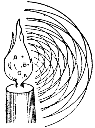
where T is the **period**, l the **length** of the pendulum and g the **gravitational acceleration**.

Huygens pendulum clock

J'ai donc montré de quelle façon l'on peut concevoir que la lumière s'étend successivement par des ondes sphériques, et comment il est possible que cette extension se fasse avec une aussi grande vitesse, que les expériences et les observations célestes la demandent. Où il faut encore remarquer que, quoique les parties de l'éther soient supposées dans un continuel mouvement (car il y a bien des raisons pour cela), la propagation successive des ondes n'en saurait être empêchée, parce qu'elle ne consiste point dans le transport de ces parties, mais seulement dans un petit ébranlement, qu'elles ne peuvent s'empêcher de communiquer à celles qui les environnent, nonobstant tout le mouvement qui les agite et fait changer de place entre elles.

Mais il faut considérer encore plus particulièrement l'origine de ces ondes et la manière dont elles s'étendent. Et premièrement, il s'ensuit de ce qui a été dit de la production de la lumière, que chaque petit endroit d'un corps lumineux, comme le Soleil, une chandelle, ou un charbon ardent, engendre ses ondes, dont cet endroit est le centre. Ainsi dans la flamme d'une chandelle (Fig. 4), étant distingués les points A, B, C, les cercles concentriques décrits autour de chacun de ces points représentent les ondes qui en proviennent. Et il en faut concevoir de même autour de chaque point de la surface et d'une partie du dedans de cette flamme.

Fig. 4.



Mais comme les percussions au centre de ces ondes n'ont point de suite réglée, aussi ne faut-il pas s'imaginer que les ondes mêmes s'entresuivent par des distances égales; et si ces distances paraissent telles dans cette figure, c'est plutôt pour marquer le progrès d'une même onde en des temps égaux, que pour en représenter plusieurs provenues d'un même centre.

Il ne faut pas au reste que cette prodigieuse

History of waves: Bernoulli family

Jakob Bernoulli (1655-1705) and Johann Bernoulli (1667-1748) were Swiss mathematicians known for their contributions to infinitesimal calculus. They were among the first mathematicians to study, understand and apply calculus to various physical problems. The book Analyse des Infiniment Petits pour l'Intelligence des Lignes Courbes written by Guillaume de l'Hôpital, which mainly consisted of the work of Johann Bernoulli (e.g. l'Hôpital rule).

Daniel Bernoulli (1700-1782), son of Johann Bernoulli. His chief work is Hydrodynamica (1738), in which all the results are consequences of the conservation of energy (Bernoulli's principle). It was known that a moving body transforms kinetic energy into potential energy when it gains height. Daniel realized that also a moving fluid exchanges kinetic energy for pressure. Mathematically this law is written as

$$\frac{1}{2}\rho|u|^2 + P = \text{constant},$$

where P is the pressure, ρ is the density and u is the velocity of the fluid.

He won 10 times the Grand Prize of the Paris Academy, for topics on Newton's theory of tides (jointly with Leonard Euler in 1740), magnetism, ocean currents.

Daniel Bernoulli also wrote some papers on vibrating strings. The history of spectral modeling of sound begins with him, who first believed that any acoustic vibration could be expressed as a superposition of sinusoidal vibrations. He showed this for several identical masses interconnected to form a discrete approximation to an ideal string.

History of waves: Isaac Newton

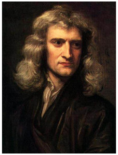
Isaac Newton (1642-1727) was an English physicist, mathematician, astronomer, natural philosopher, alchemist and theologian.

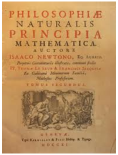
In his book Philosophiae Naturalis Principia Mathematica (1687) described:

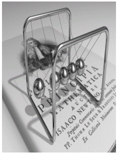
- Newton's law of universal gravitation. Two points of masses m_1 and m_2 are attracted with a force F directly proportional to the product of their masses and inversely proportional to the square of the distance d between them: $F = Gm_1m_2/d^2$, with G the gravitational constant.
- Newton's three laws of motion. • The velocity of a body remains constant unless the body is acted by external forces. • The acceleration a of a body is parallel and directly proportional to the force F and inversely proportional to the mass m, i.e. $F = ma$. • The mutual forces of action and reaction between two bodies are equal, opposite and collinear.

In the Preface of Principia, Newton says:
"I wish we could derive the rest of phenomena of nature by the same kind of reasoning from mechanical principles; for I am induced by many reasons to suspect that they may all depend upon certain forces by which the particles of bodies, by some causes hitherto unknown, are either mutually impelled towards each other, and cohere in regular figures, or are repelled and recede from each other."

In the General Scholium of Book III of Principia, Newton says:
"This most beautiful system of the sun, planets and comets could only proceed from the counsel and dominion of an intelligent and powerful being."



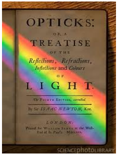




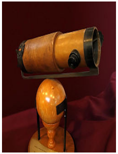
Newton's cradle

History of waves: Isaac Newton

In 1704, he published a record of experiments and deductions made from them entitled Opticks written in English (not in Latin as Principia). He considered light to be made up of extremely subtle corpuscles and ordinary matter to be made of grosser corpuscles.

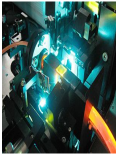


Opticks is based on his lectures on Optics between 1670 and 1672, when he investigated the dispersion of light (called *inflexion of light*), demonstrating that a prism could decompose white light into a spectrum of colours and that a lens and a second prism could recompose the multicoloured spectrum into white light.



Newton's telescope

Multi-prism dispersion theory. Newton's contribution to prismatic dispersion became central to the design of the tunable lasers more than 275 years later (with applications in astronomy, atomic separation, medicine, spectroscopy).



Tunable laser

Newton's theory of colour. The coloured light does not change its properties by separating out a coloured beam and shining it on various objects.

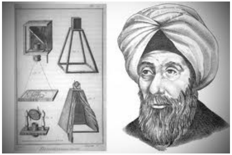
Newtonian telescope. From his work on optics, he concluded that the lens of any refracting telescope would suffer from the chromatic aberration, i.e. the dispersion of light into colours. He constructed the first known functional reflecting telescope (1672) using a mirror as objective.

History of waves: Alhazen and Sahl

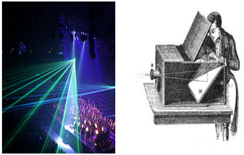
Ibn al-Haytham or Alhazen (965-1040) was a Muslim scientist. His Book of Optics (1021) has been ranked as one of the most influential books in the history of physics.

- He was the first in proving that rays of light travel in straight lines,
- in reducing reflected and refracted light rays into vertical and horizontal walls.
- He is credited with the invention of the camera obscura and pinhole camera.
- Wrote on the atmospheric refraction due to morning and evening twilight,
- on the dispersion of light into its constituent colours,
- on the finite speed of light,
- on the slower movement of light in denser bodies.
- Is the first to describe accurately the various parts of the eye and give a scientific explanation of the process of vision.

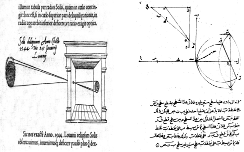
The law of refraction of light was first discovered by Ibn Sahl (940-1000), Iranian mathematician, physicist and engineer, in his treatise On burning mirrors and lenses sets (984), presenting his understanding of how curved mirrors and lenses bend and focus light.



Alhazen



Light rays propagating straight and camera obscura



Pinhole camera and Sahl manuscript

History of waves: Snell and Descartes

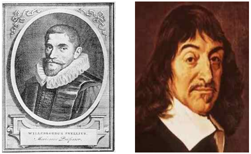
Willebrord Snel van Royen (1580-1626) - Dutch astronomer and mathematician. His name has been associated for several centuries to the discovery in 1621 of the law of refraction of light.

René Descartes (1596-1650) - French philosopher, mathematician and writer, who spent most of his life in Netherlands, considered the father of modern philosophy.

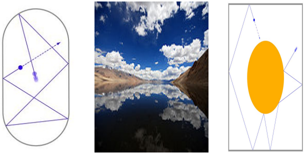
- The Cartesian coordinate system was named after him.
- He is considered the father of analytical geometry.
- He invented the convention of representing unknowns in equations by x, y and z and the coefficients by a, b and c. Also the standard notation of superscripts for powers.
- His work is the basis for the infinitesimal calculus developed by Newton and Leibniz.
- He unifies calculus and geometry by writing equations for straight lines or circles.
- In Principles of Philosophy (1644), he discovered an early form of the law of conservation of mechanical momentum.

Contributions to Optics:

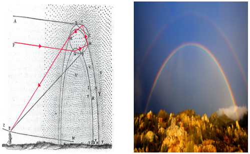
- law of refraction (Descartes's or Snell's law) - the angle subtended at the eye by the edge of the rainbow and the ray passing from the sun through the rainbow's centre is 42°.
- law of reflection in the appendix La Dioptrique of his Le discours de la méthode pour bien conduire sa raison, et chercher la vérité dans les sciences (1637).



Snell and Descartes



Bunimovich stadium, specular reflection and Sinai billiard



Double rainbow and its explanation by Descartes

History of waves: Fermat and de la Chambre

Pierre de Fermat (1601-1665) - French (of Basque origin) lawyer at the Parlement of Toulouse and an amateur mathematician, well-known for early developments leading to infinitesimal calculus. In particular, he is recognized for:

- In Methodus ad disquirendam maximam et minima (1638) and in De tangentibus linearum curvarum (1638), Fermat developed an original method for determining maxima, minima and tangents to various curves, equivalent to differentiation, and a technique to find centers of gravity for various plane and solid figures, which led to his further work on quadrature.
- research in number theory (e.g. Fermat's Last Theorem, which he described in a note at the margin of a copy of Diophantus Arithmetica).
- Shortest distance? In 1657, Marin Cureau de la Chambre (1594-1669) wrote a book entitled Light, in which explained the equality between the incidence and refraction by the fact that light takes the shortest distance. However, refraction violated this law of shortest distance. De la Chambre attributed this contradiction to all that pesky material in the medium preventing light from having liberty to move on the shortest distance.
- Fermat's principle or principle of least time (used to derive Snell's law in 1657) was the first variational principle enunciated in physics since Heron de Alexandria described the principle of least distance. It was stated by Fermat in a letter to Cureau de la Chambre in 1662, entitled Analysis of refraction and Syntesis of refraction. Unable to accept the idea that light travels faster in a denser medium as Descartes stated, Fermat affirmed that

  A ray of light propagating between two points takes the path that can be traversed in the least time.

  Fermat found the problem too difficult to be analyzed (I admit that this problem is not one of the easiest). The proof of Fermat's principle was performed by Leibniz (1684).

Fermat

de la Chambre

J. Longuski, The 7 secrets of how to think like a rocket scientist, Springer, 2007.

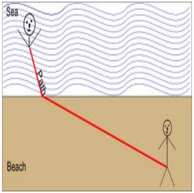
It seems that Mother Nature is an optimizer herself. Nature finds the shortest time path for a beam of light traveling through air, water and glass. Because light travels slower in glass, it avoids spending too much time in the glass and takes a more distant path through the air where it travels faster (than it does in glass). This is Fermat's principle of optics. It is the same strategy a life guard follows to save a drowning man. If the drowning man is located on a diagonal path across the water, she will run as fast as she can along the beach and only enters the water at a point where she will get to the drowning man the quickest. She knows that she swims a lot slower than she can run. She is an optimizer.

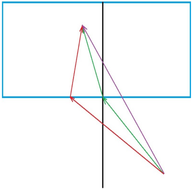
M. Cureau de la Chambre, Light, 1657.

It is a truth that has been drawn from experience, and which no one has ever contested, that the reflected ray comes off of an opaque body in the same proportion that it fell upon it...This is not a privilege of light; for not only do bodies reflect thus, but also sound and heat...This is clear for sound, because an echo, which is nothing other than a reflected sound, can only be heard in the location makes an equal angle with the first impression that the sound makes on a body...This being assumed, we must seek out the reason why reflections are made with equal angles... Indeed, could anything be said that would be more in conformity with reason than when they certify that the equality of angles occurring in reflection is made in accordance with the laws that nature maintains in all of her movements? For as she employs the shortest means in all her actions, she moves things through the shortest space: whence it comes that all bodies go directly to their centers, and that weights descend downwards, and lighter bodies ascend in straight lines, because these lines are the shortest of all, in such a way that it is necessary according to this rule that reflection be made along the shortest lines. Now it is assured that these lines make equal angles, and that if, impossibly, the angles were not equal, these lines would not be the shortest.

Fermat's principle

You are on the beach and see someone drowning and if you are a good swimmer you rush to help him. On which path you get there as fast as possible?

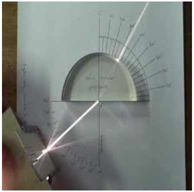


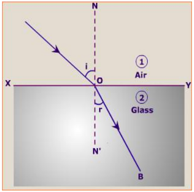


Possibilities:

- the shortest path, but you know that **you run much faster on the beach than swim in the water**.
- You go farther on the beach and jump in the water closer to the drowning man. If you go on the beach much far away you loose time.
- the fastest path. There is a best point to which you run and after that you swim. It depends on how fast you can run on the beach and swim in the water.

REFRACTION: Light takes the fastest path. It travels faster in the air than in the water.





Who optimizes? Clerselier-Fermat controversy

"The principle upon which you build your proof, namely that nature always acts by the shortest and simplest ways, is a moral principle, not a physical one, which is not and cannot be the the cause of any effect of nature...it is not by this principle that it acts, but by the secret force and virtue which lies in every thing...And it cannot be, otherwise we would be assuming some kind of awareness in nature." (Clerselier's letter to Fermat, 1662)

Interpretation. Nature acts without forethought, without choice, it does not look ahead and it is never faced with choices. It does not pick its way among several possibilities, taking into account their consequences, far or near into the future; at any time, it finds just one door open and it goes through that door. (Ekeland, The best of all possible worlds)

"Nature has obscure and hidden ways, which I have never tried to penetrate. I had only offered it some slight geometrical help in the matter of refraction, in case it had needed it...I heartily abandon you my pretended conquest in physics, provided you leave me in possession of my geometrical problem, all pure and in abstracto, by which one can find the path of a moving object which crosses two different mediums, and which tries to end its motion as soon as possible." (Fermat's letter to Clerselier, 1662)

Interpretation. Fermat associated a mathematical problem with the physical phenomenon of refraction. Clerselier objected that there is no reasonable meaning to be attached to the model: things cannot actually work that way, it cannot be that light has both the desire to travel fastest and the means to compute the quickest path. Fermat answered that light propagates as if it had both that desire and these means, and while the mathematical problem may not be an accurate description of what is happening at some deeper level of reality, it is good enough to make predictions which turn out to be in agreement with experiments. The model should be kept as an working tool for scientists until it is discarded for a better one, and the question of why it works and what it means should be left to philosophers to worry about. (Ekeland, The best of all possible worlds)

Who optimizes?

God has created the world with a definite purpose in mind, namely to make it as perfect as possible:

"This is the best of all possible worlds...God has chosen the most perfect world, that is, the one which is at the same time the simplest in hypotheses and the richest in phenomena, as might be a line in gemoetry whose construction is easy and whose properties and effects are extremely remarkable and widespread." (Leibniz, Discourse on Metaphysics, 1686)

Einstein-Bohr controversy about the foundations of quantum mechanics:

"God doesn't play dice with the world." (Einstein, Einstein and the Poet, 1943)

"I don't know, all I am saying is that, using quantum mechanics and probability theory, I can make very accurate predictions." (Bohr)

"The least action principle and with it all the minimum principles that one encounters in mechanics simply express that, in every case, whatever happens the circumstances determine uniquely." (Mach, 1883)

Gottfried Wilhelm Leibniz (1646-1716) was a German mathematician and philosopher.

- He is the first to explicitly employ the notion of function to denote any of several geometric concepts derived from a curve, such as abscissa, ordinate or tangent.
- Leibniz was the first to see that the coefficients of a system of linear equations could be arranged into a matrix and to find the solution of the system by Gaussian elimination.
- Leibniz is credited, along with Newton, with the invention of infinitesimal calculus (containing both differential and integral calculus). A critical breakthrough occurred in 1675, when he employed integral calculus for the first time to find the area under the graph of a function. The product rule in differential calculus is called Leibniz's rule. The theorem of differentiation under the integral is called the Leibniz's integral rule.
- Several celebrated mathematicians also agreed with Fermat's principle, particularly Leibniz, who gave the problem an elegant mathematical analysis (1684).

The proof by Leibniz

E. Hairer, G. Wanner, Analysis by its history, Springer, 2008

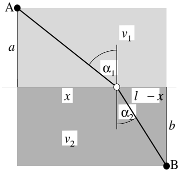
Problem: explain the law of refraction of light between two media in which the velocities are v_1 and v_2.

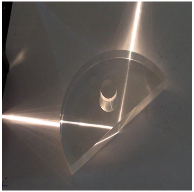
- Let A, B be two given points, one in each medium.
- We find the angles α_1 and α_2 s.t. the light travels from A to B in minimal time.
- The definition of the time function $T(x) := \frac{\sqrt{a^2+x^2}}{v_1} + \frac{\sqrt{b^2+(l-x)^2}}{v_2}$.
- Leibniz idea: Find x s.t. $T'(x) = 0$, where

$$T'(x) = \frac{1}{v_1}\frac{x}{\sqrt{a^2+x^2}} - \frac{1}{v_2}\frac{l-x}{\sqrt{b^2+(l-x)^2}}.$$

- Observe that $\sin(\alpha_1) = \frac{x}{\sqrt{a^2+x^2}}$ and $\sin(\alpha_2) = \frac{l-x}{\sqrt{b^2+(l-x)^2}}$.
- Then $T'(x) = 0$ is equivalent to Snell's law: $\frac{\sin(\alpha_1)}{v_1} = \frac{\sin(\alpha_2)}{v_2}$.
- The fact that $T''(x) > 0$ shows that x is really a minimum for T:

$$T''(x) = \frac{1}{v_1}\frac{a^2}{(a^2+x^2)^{3/2}} + \frac{1}{v_2}\frac{b^2}{(b^2+(l-x)^2)^{3/2}} > 0.$$





Obs1. The Snell's law can be written as: $\sin(\alpha_2) = v_2 \frac{\sin(\alpha_1)}{v_1}$.

Obs2. Total reflection. When $v_1 < v_2$, sometimes there is no α_2 for $\alpha_1 \in (0, \pi/2)$ s.t.

$$\left| v_2 \frac{\sin(\alpha_1)}{v_1} \right| > 1.$$

The principle of least action is the basic variational principle for dynamical systems, whose true trajectories are found by computing the action (i.e. a functional depending of all possible trajectories) and selecting the trajectory minimizing the action. The formulation of this principle is often given to Pierre-Louis Moreau de Maupertuis (1698-1759), who wrote about it:

Maupertuis, Accord de différentes lois de la nature qui avaient jusqu'ici paru incompatibles, 1744.

Maupertuis, Le lois de mouvement et du repos, déduites d'un principe de métaphysique, 1746.

"After meditating deeply on this topic, it occurred to me that light, upon passing from one medium to another, has to make a choice, whether to follow the path of shortest distance (the straight line) or the path of least time. But why should it prefer time over space? Light cannot travel both paths at once, yet how does it decide to take one path over another? Rather than taking either of these paths per se, light takes the path that offers a real advantage:

Light takes the path that minimizes its action.

The action depends on the speed of the body and on the distance it travels. However, it is neither the speed nor the distance taken separately; rather, it is proportional to the sum of the distances traveled multiplied each by the speed at which they were traveled. This action is the true expense of Nature, which she manages to make as small as possible in the motion of light."

The original definition of action by Maupertuis is the function

$$A(x) := v_1\sqrt{a^2 + x^2} + v_2\sqrt{b^2 + (l - x)^2},$$

whose minimization gives again Snell's law.

A burning mirror is a large convex lens that can concentrate the Sun's rays onto a small area, heating up it and thus resulting in ignition of the exposed surface.

The technology of the burning mirrors has been known since antiquity. Vases filled with water used to start fires were known in the ancient world and metaphorical significance was drawn (by the early Church Fathers, for instance) from the fact that the water remained cool even though the light passing through it would set materials on fire. Burning lenses were used to cauterise wounds and to light sacred fires in temples. The perpetual sacred fire in the classic temples as the Olympic torch had to be pure and to come directly from the gods. For this they used the sun's rays focused with mirrors or lenses.

Archimedes (287-212 BC) was said to have used a burning mirrors (or more likely a large number of angled hexagonal mirrors) as a weapon in 212 BC, when Syracuse was besieged by Marcus Claudius Marcellus. The Roman fleet was supposedly incinerated, though eventually the city was taken and Archimedes was slain.