Wave propagation and discontinuous Garlerkin approximations[1]

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[1] Based on A. Maricas and E. Z., Symmetric discontinuous Galerkin approximations of $1-d$ waves: Fourier analysis, propagation, observability and applications, Springer Briefs in Mathematics, 2014, 114 pp

Motivation: Boundary observation and control of the wave equation

The Cauchy problem for the $1-d$ wave equation:

$$\begin{cases} u_{tt}(x,t) - u_{xx}(x,t) = 0, & x \in \mathbf{R}, t > 0 \\ u(x,0) = u^0(x), u_t(x,0) = u^1(x), & x \in \mathbf{R}. \end{cases} \tag{1}$$

(1) is well posed in the energy space $\dot{H}^1 \times L^2(\mathbf{R})$.
The energy is constant in time:

$$E(u^0, u^1) = \frac{1}{2} \int_{\mathbf{R}} \left(|u_x(x,t)|^2 + |u_t(x,t)|^2 \right) dx. \tag{2}$$

The energy concentrated in $\mathbf{R} \setminus (-1, 1)$,

$$E_{\mathbf{R}\setminus(-1,1)}(u^0, u^1, t) = \frac{1}{2} \int_{|x|>1} \left(|u_x(x,t)|^2 + |u_t(x,t)|^2\right) dx \tag{3}$$

suffices to "observe" the total energy if $T > 2$ (characteristic time).
More precisely, for all $T > 2$ there exists $C(T) > 0$ such that

$$\boxed{E(u^0, u^1) \leqslant C(T) \int_0^T E_{\mathbf{R}\setminus(-1,1)}(u^0, u^1, t)\, dt,} \tag{4}$$

for all initial data (u^0, u^1) of finite energy.
Aplications: boundary control, stabilization, inverse problems...

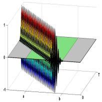


Figure: The energy of solutions propagates along characteristics that enter the observation zone in a time at most $T = 2$.

Objective

Analyze this property under **numerical discretizations**. Actually, it is by now well known that, **for classical finite-difference and finite-element discretizations, the observation constant diverges** because of the presence of high frequency spurious numerical solutions for which the *group velocity vanishes*.

In this work:

- We perform the **Fourier analysis of the Discontinuous Galerkin Methods** for the wave equation.
- We show that the **same negative results** have to be expected.
- We perform a **gaussian beam construction** showing the existence of exponentially concentrated waves, yielding, effectively, negative results.
- Our analysis indicates how **filtering techniques** should be designed to avoid these unstabilities.

See [E. Z., SIAM Review, 2005] for basic results in this field.

Finite-difference space semi-discretization:

$$\begin{cases} u_j''(t) - \frac{u_{j+1}(t)-2u_j(t)+u_{j-1}(t)}{h^2} = 0, & j \in \mathbf{Z}, t > 0 \\ u_j(0) = u_j^0, u_j'(0) = u_j^1, & j \in \mathbf{Z}. \end{cases} \tag{5}$$

For $(u_j^0, u_j^1) \in \dot{h}^1 \times \ell^2$, the discrete energy

$$E_h(u^0, u^1) = \frac{h}{2} \sum_{j \in \mathbf{Z}} \left(|D_h^1 u_j(t)|^2 + |u_j'(t)|^2 \right), \tag{6}$$

is constant in time.

But

$$\inf_{E_h(u^0,u^1)=1} \int_0^T E_{h,\mathbf{R}\setminus(-1,1)}(u^0, u^1, t)\, dt \to 0, \text{ when } h \to 0.$$

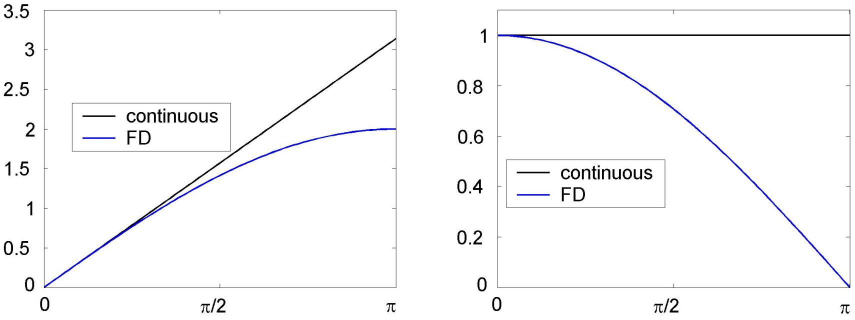


Figure: Dispertion relation (left) and group velocity (right).

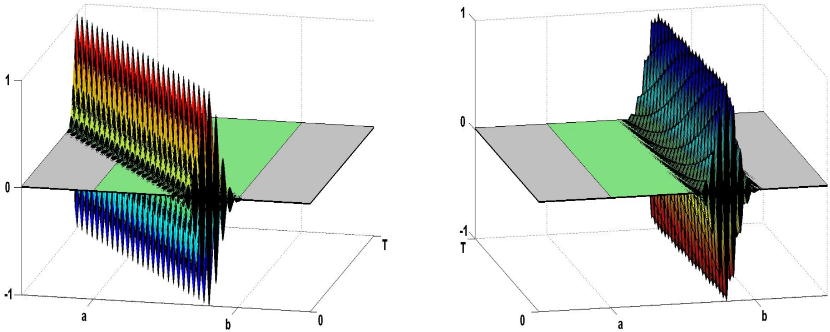


Figure: Localized waves travelling at velocity $= 1$ for the continuous wave equation (left) and wave packet travelling at very low group velocity for the FD scheme (right).

Extensive literature: Reed, W.H. & Hill, 1973; Arnold, D.N., 1979; Cockburn B., Shu C-W, 90's ; Arnold D.N., Brezzi F., Cockburn B., Marini D. 2000 - 2002,...

We consider the simplest version for the 1D wave equation in a uniform grid of size $h > 0$: $x_i = hi$.

Deformations are now piecewise linear but not necessarily continuous on the mesh points:

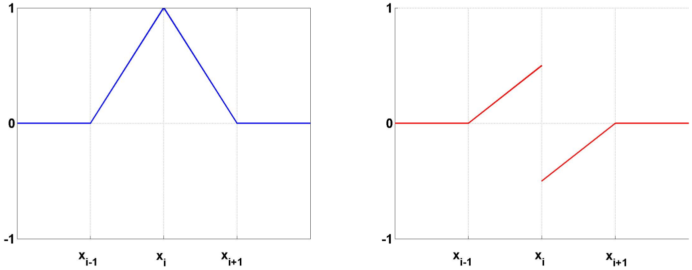


Figure: Basis funtions: ϕ_i (left) and $\widetilde{\phi}_i$ (right)

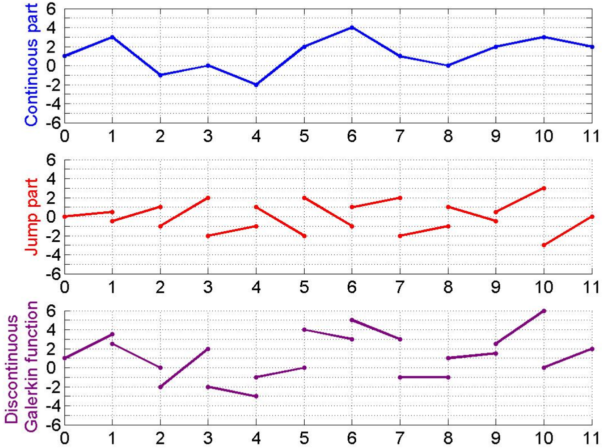


Figure: Decomposition of a DG defomration into its continuous and jump components.

Variational formulation

Relevant notation:

- Average: $\{f\}(x_i) = \frac{f(x_i+)+f(x_i-)}{2}$
- Jump: $[f](x_i) = f(x_i-) - f(x_i+)$
- $V_h = \{v \in L^2(\mathbf{R}) \big| v|_{(x_j,x_{j+1})} \in P_1, \|v\|_h < \infty\}$,
- $\|v\|_h^2 = \sum_{j\in\mathbf{Z}} \int_{x_j}^{x_{j+1}} |v_x|^2\, dx + \frac{1}{h} \sum_{j\in\mathbf{Z}} [v]^2(x_j)$

The bilinear form and the DG Cauchy problem:

$$
\begin{aligned}
a_h^s(u,v) &= \sum_{j\in\mathbf{Z}} \int_{x_j}^{x_{j+1}} u_x v_x\, dx - \sum_{j\in\mathbf{Z}} \left([u](x_j)\{v_x\}(x_j) + [v](x_j)\{u_x\}(x_j)\right) \\
&\quad + \frac{s}{h} \sum_{j\in\mathbf{Z}} [u](x_j)[v](x_j),
\end{aligned}
$$

$s > 0$ is a penalty parameter.

$$
\begin{cases}
u_h^s(x,t) \in V_h, t > 0 \\
\frac{d^2}{dt^2} \int_{\mathbf{R}} u_h^s(x,t) v(x)\, dx + a_h^s(u_h^s(\cdot,t), v) = 0, \forall v \in V_h, \\
u_h^s(x,0) = u_h^0(x), u_{h,t}^s(x,0) = u_h^1(x) \in V_h.
\end{cases}
\tag{7}
$$

DG as a system of ODE's

Decompose solutions into the FE+jump components:

$$u_h^s(x,t) = \sum_{j\in\mathbf{Z}} u_j(t)\phi_j(x) + \sum_{j\in\mathbf{Z}} \widetilde{u}_j(t)\widetilde{\phi}_j(x).$$

Then $U_h^s(t) = (u_j(t), \widetilde{u}_j(t))'_{j\in\mathbf{Z}}$ solves the system of ODE's:

$$M_h\ddot{U}_h^s(t) = R_h^s U_h^s.$$

M_h: mass matrix, R_h^s -rigidity matrix (symmetric, bloc tri-diagonal)

Applying the Fourier transform

$$\begin{pmatrix} \widehat{u}_{tt}^h(\xi,t) \\ \widehat{\widetilde{u}}_{tt}^h(\xi,t) \end{pmatrix} = -A_h^s(\xi)\begin{pmatrix} \widehat{u}^h(\xi,t) \\ \widehat{\widetilde{u}}^h(\xi,t) \end{pmatrix}. \tag{8}$$

The eigenvalues of $A_h^s(\xi)$ constitute two branches

$$\begin{cases} \Lambda^s_{p,h}(\xi) = \left(\lambda^s_{p,h}(\xi)\right)^2 & \text{(physical dispersion)} \\ \Lambda^s_{s,h}(\xi) = \left(\lambda^s_{s,h}(\xi)\right)^2 & \text{(spurious dispersion)} \end{cases}$$

The corresponding eigenvectors have the energy polarized either in the FE subspace (physical solutions) or in the jump subspace (spurious solutions).

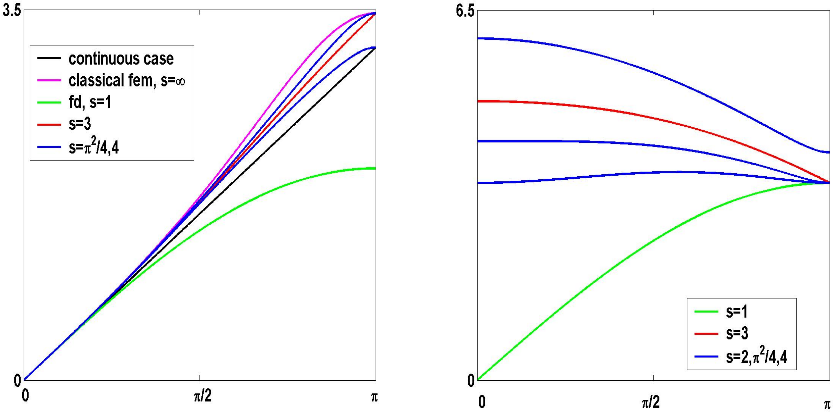


Figure: Dispersion relations for the physical (left) and the spurious (right) components for various values of the penalty parameter s.

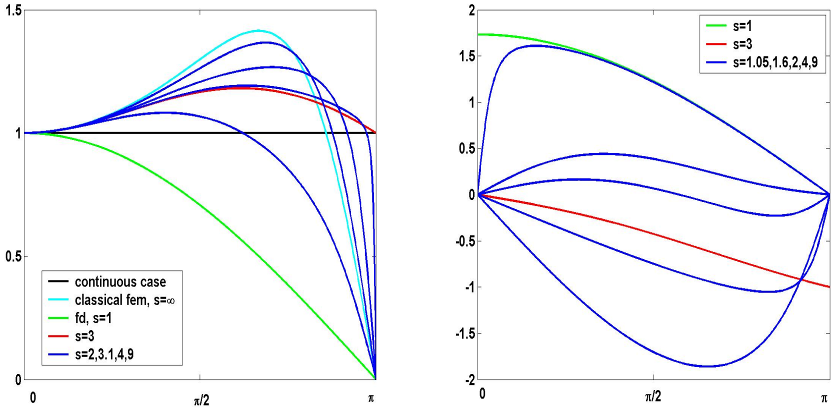
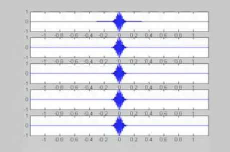
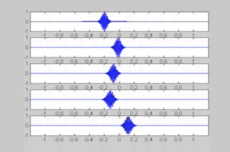
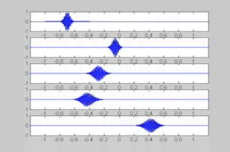


Figure: group velocity of the physical component (left) and the spurious one (right)







Conclusions

- DG provides a rich class of schemes allowing to regulate the physical components of the system, using the penalty parameter s, to fit better the behavior of the continuous wave equation.
- Despite of this, these schemes generate high frequency spurious oscillations which behave badly, generating possibly wave paquets travelling in the wrong sense.
- Further work is needed to investigate if preconditioning and/or posprocessing can remove the spurious components.