Heat Kernels

Enrique Zuazua

AU - AvH
enrique.zuazua@fau.de

April 2, 2020

HEAT KERNELS

Solve the heat equation in the whole space with initial datum $f = f(x)$:

$$u(x,t) = [G(\cdot,t) \star f(\cdot)](x)$$

where $G = G(x,t)$ is the gaussian heat kernel:

$$G(x,t) = (4\pi t)^{-N/2} \exp(-|x|^2/4t).$$

If $|x|^{k+1} \in L^p(\mathbf{R}^N)$ and $1 \leq p < N/(N-1)$ there exist $F_\alpha \in L^p(\mathbf{R}^N)$ such that

$$f = \sum_{|\alpha| \leq k} \frac{(-1)^{|\alpha|}}{\alpha!} \left(\int f(x) x^\alpha dx \right) D^\alpha \delta + \sum_{|\alpha| \leq k+1} D^\alpha F_\alpha.$$

Remark: Compare with the Taylor expansion of the Fourier transform of f.

Applying Young's inequality in this decomposition we get:

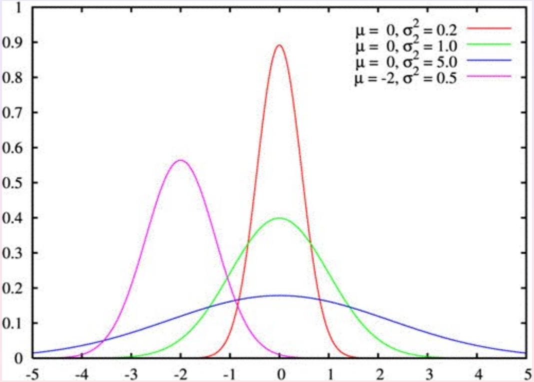
$$||u(t) - \sum_{|\alpha|\leq k} \frac{(-1)^{|\alpha|}}{\alpha!}\left(\int f(x)x^{\alpha}dx\right)D^{\alpha}G||_q$$

$$\leq Ct^{-\frac{N}{2}((k+1)N+1/p-1/q)}|||x|^{k+1}f||_p,$$

if $1 \leq p < N/(N-1)$ and $q \geq p$.
This means:

$$u(t) \sim \sum_{|\alpha|\leq k} \frac{(-1)^{|\alpha|}}{\alpha!}\left(\int f(x)x^{\alpha}dx\right)D^{\alpha}G, \quad \text{as } t \to \infty.$$

The asymptotic behavior of solutions is determined by the momenta of the initial data and the derivatives of the gaussian profile.



An example:

$$f = (\int f)\delta + \text{ div } (F), \quad \text{if } 1 \leq p < N/(N-1);$$

$$f = \text{ div } (F), \quad \text{if } N/(N-1) < p.$$

Idea of the proof:

$$< f - \int f dx \delta, \varphi > = \int f(x) \int_0^1 x \cdot \nabla \varphi(tx) dt dx$$

$$= - \int \varphi(x) \text{ div } (x \int_0^1 f(x/t) t^{-(N+1)} dt) dx.$$

Note that these identities are the dual version of the classical Hardy inequality.

For instance, $f = \text{ div } (F)$, with $||F||_p \leq C_p||xf||_p$

$$|\int f\varphi dx| = |-\int F\cdot\nabla\varphi dx| \leq C||xf||_p||\varphi||_{p'}$$

for all f and this implies that

$$||\frac{\varphi}{|x|}||_{p'} \leq C_p||\nabla\varphi||_{p'}.$$

When $p > N/(N-1)$ then $1 \leq p' < N$.

J. DUOANDIKOETXEA & E. ZUAZUA. Moments, masses de Dirac et décomposition de fonctions C. R. Acad. Sci. Paris, 315(6). 693-698. 1992.