Control of PDEs involving non-local terms

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Why nonlocal?

- Strictly speaking relevant models in Continuum Mechanics, Math Physics and Biology are of nonlocal nature:
  - Boltzmann equations in gas dynamics;
  - Navier-Stokes equations in Fluid Mechanics;
  - Keller-Segel model for Chemotaxis.
- Here, however, we shall deal with other non-local effects, due to anomalous dispersion and diffusion and memory terms. This leads to PDE involving nonlocal terms in the form of integrals either in space or time or both.
- In that setting, of course, classical PDE theory fails because of non-locality. Yet many of the existing techniques can be tuned and adapted, although this is often a delicate matter because modern PDE analysis is based on the use of localisation arguments (test and cut-off functions) that do not apply in a straightforward manner in the nonlocal context.

Goal

Try to develop a systematic analysis of the control theoretical consequences of the possible presence of non-local terms in the model.

We do it for the following model cases:

- Viscoelasticity
- Models involving memory terms
- Fractional Laplacian
- Lower order space-like nonlocal terms.
- Fractional time derivatives

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Viscoelasticity

A wave equation with both viscous Kelvin-Voigt damping:

$$
\begin{aligned}
y_{tt} - \Delta y - \Delta y_t &= 1_\omega h, && x \in \Omega,\ t \in (0, T), && (1)\\
y &= 0, && x \in \partial\Omega,\ t \in (0, T), && (2)\\
y(x,0) = y_0(x),\ y_t(x,0) &= y_1(x) && x \in \Omega. && (3)
\end{aligned}
$$

Here, Ω is a smooth, bounded open set in $\mathbb{R}^N$ and $h = h(x,t)$ is a control located in a open subset ω of Ω.

We address the controllability problem:

Given (y_0, y_1), to find a control h such that the associated solution to (1)-(3) satisfies

$$y(T) = y_t(T) = 0.$$

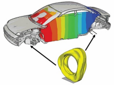
Viscoelasticity arises in areas such as **biomechanics**, **power industry** or **heavy construction**:

- Synthetic polymers;
- Wood;
- Human tissue, cartilage;
- Metals at high temperature;
- Concrete, bitumen;
- ...

Viscoelastic materials are those for which the behavior combines liquid-like and solid-like characteristics. [1]





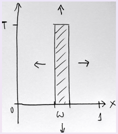



[1] See H. T. Banks, S. Hu and Z. R. Kenz, A Brief Review of Elasticity and Viscoelasticity for Solids, Adv. Appl. Math. Mech., 3 (1), (2011), 1-51.

A geometric obstruction

Standard results on unique continuation do not apply. The principal part of the operator is $\partial_t \Delta$.

Then characteristic hyperplanes are of the form $t = t_0$ and $x \cdot e = 1$.



And the zero sets do not propagate by standard unique continuation arguments.

This phenomenon was previously observed by S. Micu in the context of the Benjamin-Bona-Mahoni equation [2] [3]

In that context the underlying operator is

$$\partial_t - \partial^3_{xxt}$$

but its principal part is the same

$$\partial^3_{xxt}.$$

Viscoelasticity = Waves + Heat

$$y_{tt} - \Delta y - \Delta y_t = 0$$

$$=$$

$$y_{tt} - \Delta y = 0$$

$$+$$

$$\partial_t [y_t] - \Delta [y_t] = 0$$

Both equations are controllable. Should then the superposition be controllable as well?

The delicate role of splitting and alternating directions in the controllability of PDE:

$$x' + A_1 x + A_2 x = Bu$$

Viscoelasticity = Heat + ODE

Note that

$$y_{tt} - \Delta y - \Delta y_t + y_t = (\partial_t - \Delta)(\partial_t + I).$$

Then

$$\begin{aligned} y_t + y &= v, && (4)\\ v_t - \Delta v &= 1_\omega h, && (5)\\ v(x,t) = y(x,t) &= 0, \qquad (x,t) \in \partial\Omega \times (0,T), && (6)\\ v(x,0) &= y_1(x) + y_0(x), \qquad x \in \Omega, && (7)\\ y(x,0) &= y_0(x), \qquad x \in \Omega. && (8) \end{aligned}$$

The question now becomes: *Given* (y_0, z_0) *to find a control* h *such that the associated solution to* (4)-(8) *satisfies*

$$y(T) = v(T) = 0.$$

The main obstruction is the presence of the ODE governing the dynamics of y, in which no diffusion, dispersion or propagation effects arise.

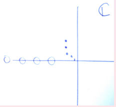
Viscoelasticity = Heat + Memory

Note that

$$y_{tt} - \Delta y - \Delta y_t = \partial_t[y_t - \Delta y - \Delta \int_0^t y].$$

The later, heat with memory, was addressed by Guerrero and Imanuvilov[4], showing that the system is not null controllable.

The spectrum contains a sequence accumulating at $\lambda = 0$ which is an obstruction for any kind of observability inequality to hold. This is precisely due to the underlying ODE component...



[4] S. Guerrero, O. Yu. Imanuvilov, Remarks on non controllability of the heat equation with memory, ESAIM: COCV, 19 (1)(2013), 288–300.

The controllability of the system is unclear:

$$v_t - \Delta v = 1_\omega h, \quad y_t + y = v + 0.$$

But we can consider the system with an added fictitious control:

$$v_t - \Delta v = 1_\omega h, \quad y_t + y = v + 1_\omega k$$

$$[y_{tt} - \Delta y - \Delta y_t + y_t = 1_\omega h + (\delta_t - \Delta)(1_\omega k)].$$

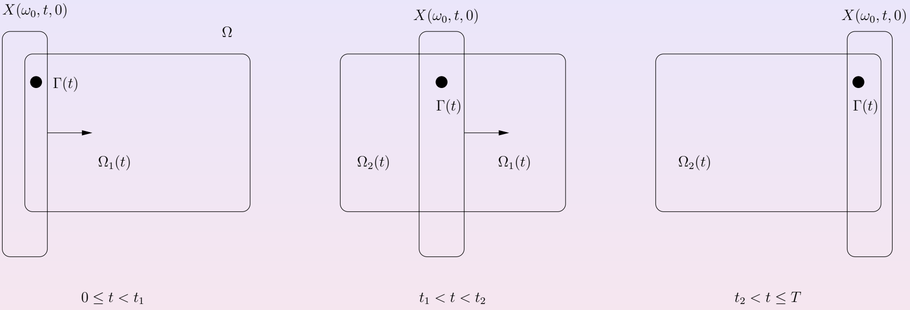
Control in two steps:

- Use the control h to control v to zero in time $T/2$.
- Then use the control k to control the ODE dynamics in the time-interval $[T/2, T]$.

Warning. The second step cannot be fulfilled since the ODE does not propagate the action of the controller which is confined in ω.

Possible solution: Make the control in the second equation move or, equivalently, replace the ODE by a transport equation.

A successful example of moving support of the control



$$
\begin{aligned}
v_t - \Delta v &= 1_{\omega(t)} h, \\
y_t + y &= v + 1_{\omega(t)} k.
\end{aligned}
\tag{9}
$$

This strategy was introduced and found to be successful in

- P. Martin, L. Rosier, P. Rouchon, Null Controllability of the Structurally Damped Wave Equation with Moving Control, SIAM J. Control Optim., 51 (1)(2013), 660–684.
- L. Rosier, B.-Y. Zhang, Unique continuation property and control for the Benjamin-Bona-Mahony equation on a periodic domain, J. Differential Equations 254 (2013), 141-178.

by using Fourier series decomposition.

If ω moves, i. e. $\omega(t)$, with a velocity field $a(t)$, then by a change of coordinates, this has the same effect as replacing the ODE by:

$$y_t + a(t) \cdot \nabla y = 1_\omega k.$$

And it is sufficient that all characteristic lines pass by ω to ensure controllability or, in other words, that the set $\omega(t)$ covers the whole domain Ω in its motion.

This strategy was proved to be successful in the multi-d context in F. Chaves, L. Rosier & E. Zuazua, Null controllability of a system of viscoelasticity with a moving control, Journal de Mathématiques Pures et Appliquées, 101 (2014) 198-222

The proof employs:

- The duality with the observability inequality
- Simultaneous Carleman inequalities for heat equations and ODEs with moving weights.

Under some technical assumptions on the support of the control....

Observability of the adjoint system

The adjoint system reads

$$-p_t - \Delta p = 0, \; -q_t + q = p.$$

And the challenge is to prove

$$\int_\Omega [|p(x,0)|^2 + |q(x,0)|^2] dx \leq C \int_0^T \int_{\omega(t)\times(0,T)} |q(x,t)|^2 \; dxdt.$$

This is done employing Carleman inequalities of the form

$$\int_Q \rho^{-2}[p^2 + q^2] \, dx \, dt \leq C \int_{\omega(t)\times(0,T)} \rho^{-2} q^2 \, dx \, dt,$$

and the difficulty is to find a weight valid simultaneously for the ODE (or transport equation) and the heat equation.

The philosophy: Local information on q leads to local information on p by reading-off the ODE. Local information on p diffuses according to the heat equation and leads to global estimates on p. Going back to the ODE we have local information on q and complete knowledge on p. This suffices.

A failing (?) moving support: Open problem

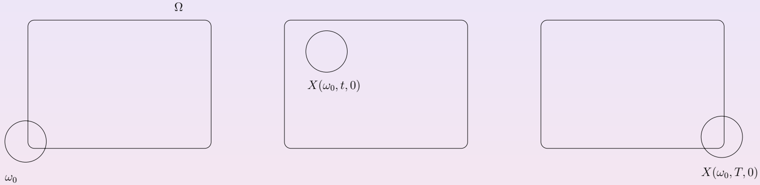


Figure: $\Omega \setminus \omega(t)$ is not split in two disjoint connected components

Other related systems

This issue of moving control is closely related to:

1. Vanishing viscosity limit for the control of convection-diffusion equations
   J. M. Coron and S. Guerrero (2005), S. Guerrero and G. Lebeau (2007), P. Lissy (2015).
2. Control of compressible Navier-Stokes equations
   S. Ervedoza, O. Glass, S. Guerrero & J.-P. Puel (2012) and D. Mitra, M. Ramaswamy and M. Renardy (2015)
3. Thermoelasticity
   - G. Lebeau, E. Zuazua, Null controllability of a system of linear thermoelasticity. ARMA, 141 (4)(1998), 297-329.
   - P. Albano, D. Tataru, Carleman estimates and boundary observability for a coupled parabolic-hyperbolic system, Electron. J. Differential Equations, 22 (2000), 1–15.

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Heat processes with memory terms

[5] A simple system of heat process with memory:

$$\begin{cases} y_t - \Delta y + \int_0^t y(s)ds = u\chi_\omega(x) & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(0) = y_0 & \text{in } \Omega. \end{cases} \tag{10}$$

Setting $z(t) = \int_0^t y(s)ds$, this system can be rewritten as

$$\begin{cases} y_t - \Delta y + z = u\chi_\omega(x) & \text{in } Q, \\ z_t = y & \text{in } Q, \\ y = z = 0 & \text{on } \Sigma, \\ y(0) = y_0,\ z(0) = 0 & \text{in } \Omega. \end{cases} \tag{11}$$

And the previous results apply.

[5] Joint work with F. Chaves-Silva & X. Zhang

More general exponential/polynomial memory kernels

$$\begin{cases} y_t - \Delta y + \int_0^t M(t-s)y(s)ds = u\chi_\omega(x) & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(0) = y_0 & \text{in } \Omega, \end{cases} \tag{12}$$

with

$$M(t) = e^{at} \sum_{k=0}^{K} a_k t^k \tag{13}$$

where $K \in \mathbb{N}$, and $a, a_0, \cdots, a_K, b_0, \cdots, b_K$ are real constants.
Writing

$$Z = \int_0^t M(s-t)y(s)ds \tag{14}$$

we get

$$\begin{cases} y_t + \Delta y = Z & \text{in } Q, \\ \partial_t^{K+1} Z = \sum_{k=0}^{K} k! a_k \partial_t^{K-k} y & \text{in } Q. \end{cases} \tag{15}$$

What about more general memory kernels?
Note, for instance, that for general analytic kernels we get a coupled PDE+ODE system involving an infinite number of ODEs.
Can a strategy in the spirit of Cauchy-Kovalewski be applied?

Waves with memory

Similar techniques can be applied to reduce the following wave equation with memory [6]

$$\begin{cases} y_{tt} - \Delta y + \int_0^t y(s)ds = \chi_O u & \text{in } Q, \\ z_t = y & \text{in } Q, \\ y = z = 0 & \text{on } \Sigma, \\ y(0) = y_0,\ y_t(0) = y_1,\ z(0) = 0 & \text{in } \Omega, \end{cases} \tag{16}$$

into

$$\begin{cases} y_{tt} - \Delta y + z = \chi_O u & \text{in } Q, \\ z_t = y & \text{in } Q, \\ y = z = 0 & \text{on } \Sigma, \end{cases} \tag{17}$$

by setting

$$z(t) = \int_0^t y(s)ds.$$

[6] joint work with Q. Lü & X. Zhang, JMPA, 2017

In view of this structure it is natural to introduce the following Moving Geometric Control Condition (MGCC):[7]

We say that an open set $U \subset (0, T) \times \Omega$ satisfies the MGCC, if

1. all rays of geometric optics of the wave equation enter into U before time T;
2. the projection of U onto the x variable covers the whole domain Ω.

This geometric condition turns out to be sufficient for moving control.

[7] Similar geometric conditions arise in the general context of waves with moving control regions as in a recent work by G. Lebeau, J. Le Rousseau, P. Terpolilli and E. Trélat.

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Introduction

[8] Controllabity for the fractional Schrödinger equation is shown

$$iu_t + (-\Delta)^s u = 0 \tag{18}$$

on a bounded $C^{1,1}$ domain $\Omega \subset \mathbb{R}^n$, provided:

- $s \geq 1/2$;
- The control is active on a neighborhood of the boundary or subset of the boundary fulfilling the classical multiplier conditions;
- In the limit case $s = 1/2$, the control time T is large enough.

As a consequence the following fractional wave equation is also controllable:

$$u_{tt} + (-\Delta)^{2s} u = 0. \tag{19}$$

Note that we do not adopt the definition of fractional Laplacian in terms of the spectrum, in which case one could use the wave-packets estimates inherited from the well-known properties of classical wave and Schrödinger operators.

[8] U. Biccari, PhD Thesis, UPV/EHU, Bilbao, Spain, Nov. 2016

Fractional Laplacian

$$(-\Delta)^s u(x) := c_{n,s} P.V. \int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{n+2s}} dy, \qquad s \in (0,1) \tag{20}$$

$$c_{n,s} := \frac{s2^{2s}\Gamma\left(\frac{n+2s}{2}\right)}{\pi^{n/2}\Gamma(1-s)}$$

Fractional Sobolev space

$$H^s(\Omega) := \left\{ u \in L^2(\Omega) \Big| \frac{|u(x)-u(y)|}{|x-y|^{\frac{n}{2}+s}} \in L^2(\Omega\times\Omega) \right\},$$

$$\Big| u \Big|_{H^s(\Omega)} := \left(\int_\Omega |u|^2 dx + \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}} dxdy \right)^{\frac{1}{2}}, \tag{21}$$

$$H_0^s(\Omega) := \{ u \in H^s(\mathbb{R}^n) |\, u = 0 \ in \ \Omega^c \}.$$

Observability inequality

The problem is equivalent to proving the observability inequality

$$\left| p_0 \right|^2_{L^2(\Omega)} \leq C \int_0^T \int_\omega |p|^2 dxdt \tag{22}$$

for the uncontrolled solutions:

$$\begin{cases} ip_t + (-\Delta)^s p = 0 & in \;\; \Omega \times [0, T] := Q \\ p \equiv 0 & in \;\; \Omega^c \times [0, T] \\ p(x, 0) = p_0(x) & in \;\; \Omega. \end{cases} \tag{23}$$

Pohozaev identity for the fractional Schrödinger equation

Let Ω be a bounded $C^{1,1}$ domain, $s \in (0,1)$ and $\delta(x)$ be the distance of a point x from $\partial\Omega$. Moreover, let $\Sigma := \partial\Omega \times [0,T]$. The following identity holds for sufficiently smooth solutions of the adjoint system:

$$\begin{aligned}\Gamma(1+s)^2 \int_\Sigma \left(\frac{|p|}{\delta^s}\right)^2 (x\cdot\nu)\,d\sigma\,dt \\ = 2s\int_0^T \left|(-\Delta)^{s/2} p\right|^2_{L^2(\Omega)} dt + \mathrm{Im}\int_\Omega \bar{p}(x\cdot\nabla p)\,dx\Big|_0^T\end{aligned} \tag{24}$$

where ν is the unit outward normal to $\partial\Omega$ at x and Γ is the Gamma function.

Proof

[9]

The following identity holds for the fractional Laplacian:

$$\int_\Omega (x\cdot\nabla p)(-\Delta)^s p dx = \frac{2s-n}{2}\int_\Omega p(-\Delta)^s p dx - \frac{\Gamma(1+s)^2}{2}\int_{\partial\Omega}\left(\frac{p}{\delta^s}\right)^2 (x\cdot\nu) d\sigma \tag{25}$$

where ν is the unit outward normal to $\partial\Omega$ at x and Γ is the Gamma function.

Out of this identity, one can extend it to the eigenfunctions of the fractional Laplacian and then, by Fourier series expansions, to the time-evolution problem.

[9] Xavier Ros-Oton, Joaquim Serra, The Pohozaev identity for the fractional Laplacian, Arch. Rat. Mech. Anal. 213 (2014), 587-628.

Boundary observability

From the time-dependent Pohozaev identity we can obtain the following boundary observability inequalities:

(i) If $s \in (1/2, 1)$, for any $T > 0$ it holds

$$A_1 \Big| p_0 \Big|^2_{H^s(\Omega)} \leq \int_\Sigma \left(\frac{|p|}{\delta^s}\right)^2 (x \cdot \nu) d\sigma dt \leq A_2 \Big| p_0 \Big|^2_{H^s(\Omega)} \qquad (26)$$

(ii) If $s = 1/2$, then there exists a minimal time $T_0 > 0$ such that (26) holds for any $T \geq T_0$.

- To get the interior observability inequality out of the boundary one, we need to localise estimates near the boundary.
  This turns out to be delicate in the fractional diffusion model.
- The following elementary identity (yet requiring a technical proof) is required:[10]

  Let $1/2 < s < 1$ and $\psi \in H_0^s(\Omega)$ and $\eta \in C^\infty(\mathbb{R}^N)$ be a cut-off function such that $\eta = 1$ in $\hat{\omega}$, $0 \leq \eta \leq 1$ in $\omega \setminus \hat{\omega}$ and $\eta = 0$ in ω^c, $\hat{\omega}$ being a neighbourhood of the boundary such that $(\Omega \cap \overline{\hat{\omega}}) \subset \omega$. Then

  $$(-\Delta)^s(\psi\eta) = \psi(-\Delta)^s\eta + R$$

  and

  $$\left|R\right|_{L^2(\mathbb{R}^N)} \leq C\left[\left|\eta\right|_{H^s(\omega)} + \left|\eta\right|_{L^2(\omega^c)}\right].$$

This identity is obvious for the classical Laplacian since

$$-\Delta(\psi\eta) = -\psi\Delta\eta - 2\nabla\psi\cdot\nabla\eta - \Delta\psi\eta.$$

[10] U. Biccari, M. Warma, E. Zuazua, Local elliptic regularity for the Dirichlet fractional Laplacian, Advanced Nonlinear Studies, Advanced Nonlinear Studies, 17 (2017), 387-409.

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Consider the following controlled heat equation involving nonlocal in space terms:[11]

$$\begin{cases} y_t - \Delta y + \int_\Omega K(x,\xi) y(\xi,t)\, d\xi = v 1_\omega & \text{in} \quad Q, \\ y = 0 & \text{on} \quad \Sigma, \\ y(x,0) = y^0(x) & \text{in} \quad \Omega \end{cases} \tag{27}$$

And let us analyse its null controllability property, which is equivalent to the observability inequality

$$\left|\phi(\cdot\,,0)\right|^2 \leq C \int_{\omega\times(0,T)} |\phi|^2 \, dx\, dt \qquad \forall \phi^T \in L^2(\Omega) \tag{28}$$

for the solutions of the adjoint system

$$\begin{cases} -\phi_t - \Delta \phi + \int_\Omega K(\xi,x) \phi(\xi,t)\, dt = 0 & \text{in} \quad Q, \\ \phi = 0 & \text{on} \quad \Sigma, \\ \phi(x,T) = \phi^T(x) & \text{in} \quad \Omega. \end{cases} \tag{29}$$

[11] E. Fernández-Cara, Q. Lü and E. Z, SICON, 2016, 54 (4), pp. 2009–2019.

Recall that the classical way to establish an estimate of this kind is to start from a global Carleman inequality of the form

$$\int_Q \rho^{-2}|\phi|^2 \, dx \, dt \leq C \int_{\omega\times(0,T)} \rho^{-2}|\phi|^2 \, dx \, dt, \tag{30}$$

where C is independent of ϕ^T and $\rho = \rho(x,t)$ is an appropriate weight function that blows up as $t \to T$.

Classical Carleman estimates for parabolic PDEs give

$$\begin{aligned}\int_Q \rho^{-2}|\phi|^2 \, dx \, dt \leq C(\varepsilon) \int_{\omega\times(0,T)} \rho^{-2}|\phi|^2 \, dx \, dt \\ +\varepsilon \int_{\Omega\times(0,T)} \rho^{-2}\Big|\int_\Omega K(\xi,x)\phi(\xi,t)\Big|^2 \, dx \, dt.\end{aligned} \tag{31}$$

The non-local second term in the right hand side cannot be absorbed by the left hand side.

These difficulties do not arise when dealing with classical potential terms acting locally in space, i. e. for equations of the form

$$-\phi_t - \Delta\phi + K(x,t)\phi(x,t) = 0.$$

We thus develop an alternate approach, based on Fourier analysis.

Denote by $\lambda_1, \lambda_2, \dots$ (resp. $\phi_1, \phi_2, \dots$) the eigenvalues (resp. the unit L^2 norm eigenfunctions) of the Dirichlet Laplacian in Ω. Recall that $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots$, $\lambda_m \sim m^{2/N}$ as $m \to +\infty$ and $\phi_1 > 0$ in Ω.

We impose the following conditions on the kernel $K \in L^2(\Omega \times \Omega)$:

-
$$x \mapsto \int_\Omega K(\xi, x) f(\xi)\, d\xi \text{ is analytic for all } f \in L^2(\Omega) \tag{32}$$
-
$$\begin{cases} K(x,\xi) = \sum_{m,j\geq 1} k_{mj}\phi_m(x)\phi_j(\xi) \ \text{ in } \ L^2(\Omega\times\Omega), \ \text{ with} \\ |K|_R^2 \triangleq \sum_{m\geq 1}\left(\sum_{j\geq 1}\lambda_j^{-1}|k_{mj}|^2\right)\lambda_m^{-1}e^{2R\sqrt{\lambda_m}} < +\infty, \end{cases} \tag{33}$$

where $R > 0$ sufficiently large that can be defined through the observability properties of the free equation.

In the context of the classical free heat equation, in the absence of potential terms, recall that there exist $R(\Omega, \omega, T), C(\Omega, \omega) > 0$ such that, for all $f \in L^2(\Omega)$, one has:

$$\sum_{j \geq 1} e^{-2R\sqrt{\lambda_j}} |(f, \phi_j)|^2 \leq C \int_{\omega \times (0,T)} \Big| \sum_{j \geq 1} (f, \phi_j) e^{-\lambda_j (t-T)} \phi_j(x) \Big|^2 dx\, dt.$$

This is a consequence of the global Carleman inequality (due to Fursikov & Imanuvilov and Lebeau & Robbiano) for the heat equation as observed by E. Fernández-Cara & E. Z. in ADE, 2000.

This observability inequality, is rather weak. But on the other hand it is also sharp in the sense that the observed norm cannot be better than

$$\sum_{j \geq 1} e^{-2R\sqrt{\lambda_j}} |(f, \phi_j)|^2$$

for some R.

Note however that so far the sharp constant R is not known in general although its existence is guaranteed.

This norm provides however a functional setting in which the non-local lower order term can be treated as a compact perturbation of the free dynamics.

For any $\phi^T \in L^2(\Omega)$, denote by ϕ the solution to (29) and write

$$\Phi = p + \zeta,$$

where p is the unique solution to

$$\begin{cases} -p_t - \Delta p = 0 & \text{in} \quad Q, \\ p = 0 & \text{on} \quad \Sigma, \\ p(x,T) = \phi^T(x) & \text{in} \quad . \end{cases} \tag{34}$$

and

$$\begin{cases} -\zeta_t - \Delta \zeta + \int_\Omega K(\xi,x)\zeta(\xi,t)d\xi = -\int_\Omega K(\xi,x)p(\xi,t)d\xi & \text{in} \quad Q, \\ \zeta = 0 & \text{on} \quad \Sigma, \\ \zeta(x,T) = 0 & \text{in} \quad \Omega. \end{cases} \tag{35}$$

In this functional setting (exponentially weighted Fourier norm) and under the previous analyticity assumptions on the nonlocal potential, the reminder term ζ can be shown to be a compact perturbation.
Compactness-uniqueness arguments can be developed, reducing the observability inequality for the nonlocal problem to an unique continuation problem.
Can one guarantee that the unique eigenfunction

$$-\Delta\Psi + \int_\Omega K(\xi, x)\Psi(\xi)d\xi = \lambda\Psi$$

such that

$$\Psi(x) = 0 \quad \text{in} \quad \omega$$

is the null one, $\Psi \equiv 0$?

This can be easily achieved under the assumption that the kernel K depends analytically on x.
What other results can be expected in that respect?

For the wave equation:

$$\begin{cases} y_{tt} - \Delta y + \int_\Omega K(x,\xi) y(\xi,t)\, d\xi = v 1_\omega & \text{in} \quad Q, \\ y = 0 & \text{on} \quad \Sigma, \\ y(x,0) = z^0(x), \quad y_t(x,0) = z^1(x) & \text{in} \quad . \end{cases} \tag{36}$$

the same arguments apply but, this time, milder assumptions on the Fourier coefficients of the kernel are needed since the perturbation argument can be developed in the standard energy space.

Note however that the analyticity of the kernel with respect to x is needed for unique continuation to hold.

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Consider the following control system:[12]

$$\begin{cases} \partial^{\alpha}_{t,0+} y - y = u \quad \text{in } (0, +\infty), \\ y(0) = y_0, \end{cases} \tag{37}$$

with $\alpha \in (0,1)$, $y_0 \in \mathbb{R}$ and $u \in L^2(0,T)$, with the Caputo derivative:

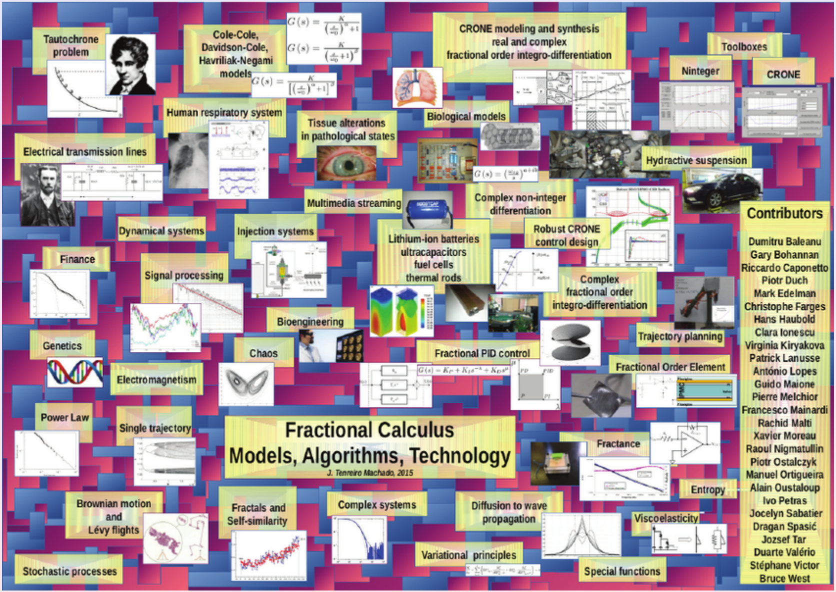
$$\partial^{\alpha}_{t,a+} f \triangleq \frac{1}{\Gamma(1-\alpha)} \int_a^t \frac{f'(s)}{(t-s)^{\alpha}} ds. \tag{38}$$

System (37) is null controllable at time $T > 0$ if for any $y_0 \in \mathbb{R}$, there is a control $u \in L^2(0,T)$ ($u(t) \equiv 0$ for all $t \geq T$) such that the corresponding solution $y(\cdot)$ satisfies that $y(t) = 0$ for all $t \geq T$.

The following holds: [13]

[12] Q. Lü, E. Zuazua, MCSS, (2016), no. 2, 28:10.

[13] Complementing earlier results by D. Matignon and B. d'Andréa-Novel, 1996.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Theorem

Whatever $T > 0$ is, system (37) is not null controllable.

Sketch of the proof.

If the system were controllable:

$$\int_0^t \frac{y'(s)}{(t-s)^\alpha}ds = \int_0^T \frac{y'(s)}{(t-s)^\alpha}ds = 0, \quad \forall\, t \geq T.$$

Taking derivatives

$$\int_0^T \frac{y'(s)}{(t-s)^{\alpha+j}}ds = 0, \quad \forall\, t > T,\ j \in \{0\} \cup \mathbb{N}. \tag{39}$$

This, together with the density of polynomials (Weierstrass approximation theorem)

$$y'(\cdot) \equiv 0 \text{ in } [0, T).$$

Taking into account that $y(T) = 0$ this also implies that $y \equiv 0$ in $[0, T]$.

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