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Dynamics and control for multi-agent networked systems

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Collective behavior models

- Describe the dynamics of a system of interacting individuals.
- Applied in a large spectrum of subjects such as collective behavior, synchronization of coupled oscillators, random networks, multi-area power grid, opinion propagation,...

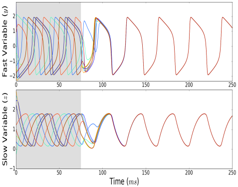

Figure: Fitz-Hugh-Nagumo oscillators [Davison et al., Allerton 2016]

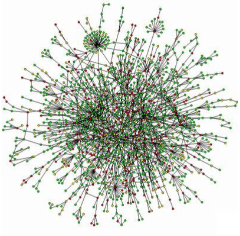

Figure: Yeast's protein interactions [Jeong et al., Nature, 2001]

Figure: German electric network

Some basic references on the Dynamics and Control on networks and graphs

[1] Kuramoto, Y. (1984). Chemical Oscillations, Waves, and Turbulence. Springer-Verlag Berlin Heidelberg.

[2] Olfati-Saber, R., Fax, J. A. & Murray, R. M. Consensus and cooperation in networked multi-agent systems. IEEE Proc. 95, 1 (2007), 215–233.

[2] Y.-Y Liu, J.-J. Slotine & A.-L. Barabási, Controllability of Complex Networks, Nature, 473, 167–173 (12 May 2011).

[3] T. Vicsek & A. Zafeiris, Collective motion, Physics Reports 517 (2012) 71–140.

[4] S. Motsch & E. Tadmor. Heterophilious dynamics enhances consensus. SIAM Review 56, 4 (2014), 577–621.

...

And many others[1] [2]

[1] M. Caponigro, M. Fornasier, B Piccoli & E. Trélat, M3AS, 2015
[2] M. Burger, R. Pinnau, A. Roth, C. Totzeck & O. Tse, arXiv 2016.

Complex behavior by simple interaction rules

Systems of Ordinary Differential Equations (ODEs) in which each agent's dynamics follows a prescribed law of interactions:

First-order consensus model

$$\dot{x}_i(t) = \frac{1}{N}\sum_{j=1}^{N} a_{i,j}(x_j(t) - x_i(t)), \quad i = 1, \dots, N$$

- It describes the opinion formation in a group of N individuals.
- $x_i \in \mathbb{R}^d$, $d \geq 1$, represents the opinion of the i-th agent.

  [J. R. P. French, A formal theory of social power, Psychol. Rev., 1956].
- It applies in several fields including information spreading of social networks, distributed decision-making systems or synchronizing sensor networks, ...

From random to consensus

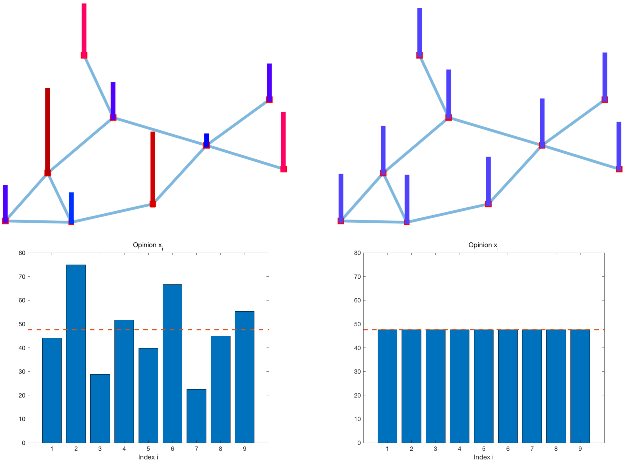

Figure: Opinions over a network : random versus consensus states

Linear versus Nonlinear

- Linear networked multi-agent models: $a_{i,j}$ are the elements of the adjacency matrix of a graph with nodes x_i

$$a_{i,j} := \begin{cases} a_{j,i} > 0, & \textit{if } i \neq j \textit{ and } x_i \textit{ is connected to } x_j \\ 0, & \textit{otherwise}. \end{cases}$$

This leads to the semi-discrete heat equation on the graph.

- Nonlinear alignment models:

$$a_{i,j} := a(|x_j - x_i|), \quad \textit{where} \quad a : \mathbb{R}_+ \to \mathbb{R}_+,$$

$a \geq 0$ is the influence function. The connectivity depends on the contrast of opinions between individuals.

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Limitation of the mean-field representation

As the number of agents $N \to \infty$, ODE $\to$ PDE.

- Nonlinear alignment models:

$$\dot{x}_i = \frac{1}{N}\sum_{j=1}^{N} a(|x_j - x_i|)(x_j - x_i), \quad i = 1, \dots, N, \ \ a : \mathbb{R}_+ \to \mathbb{R}_+.$$

Classical **mean-field theory**: Define the N-particle distribution function[3]

$$\mu^N = \mu^N(x,t) := \frac{1}{N}\sum_{i=1}^{N} \delta_{x_i(t)}.$$

and let $N \to +\infty$.

[3] P. A. Raviart, Particle approximation of first order systems, J. Comp. Math., 4 (1) (1986), 50-61.
By particle methods of approximation of time-dependent problems in PDE, we mean numerical methods where, for each time t, the exact solution is approximated by a linear combination of Dirac measures...

- The limit μ of the empirical measures μ^N solves the the nonlocal transport equation[4]

$$\partial_t \mu(x,t) = \partial_x \Big(\mu(x,t) V[\mu(x,t)] \Big)$$
$$V[\mu](x,t) := \int_{\mathbb{R}^d} a(|x-y|)(x-y)\mu(y,t)dy.$$

The convolution kernel describes the mixing of opinions by the interaction of agents along time.

- In other words:[5]

$$\partial_t \mu = \partial_x \left(\mu(x,t) \int_{\mathbb{R}^d} a(|x-y|)(x-y)\mu(y,t)\, dy \right).$$

[4] The system of ODEs describing the agents dynamics defines the characteristics of the underlying transport equation. The coupling of the agents dynamics introduces the non-local effects on trasport.

[5] Motsch and Tadmor, SIAM Rev., 2014

The mean field model does not track individuals!

- The mean-field equation involves the density μ, which does not contain the full information of the state.

- The density μ does not keep track of the identities of agents (label i).[6] Different configurations x_i with the same distribution μ

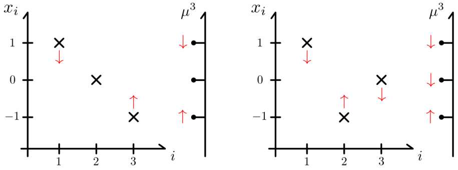

Figure: $x^1 = (-1, 0, 1)$ (left) and $x^2 = (-2, 3, -1)$ (right) generate the same density function.

[6] $\mu^N(x) := \frac{1}{N}\sum_{i=1}^{N} \delta_{x_i}$

Graph limit method: finite-difference approach

- Based on the theory of graph limits (Medvedev, SIAM J. Math. Anal., 2014).
- Considering the phase-value function $x^N(s,t)$ defined as

$$x^N(s,t) = \sum_{i=1}^{N} x_i(t)\chi_{I_i}(s,t), \quad s \in (0,1),\ t > 0,\ \bigcup_{i=1}^{N} I_i = [0,1].$$

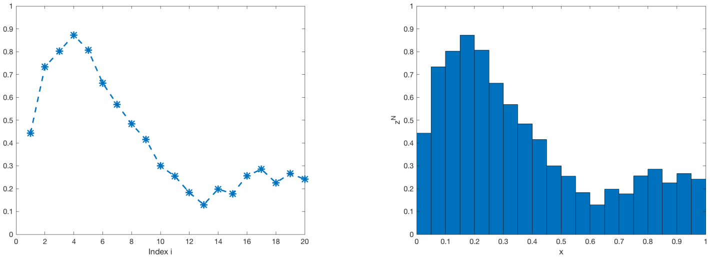

Figure: Opinion ($N = 20$) and its finite-difference function z^{20} on $[0,1]$

- Let $(x_i^N)_{i=1}^N$ be the solution of the following consensus model:

$$\dot{x}_i^N = \frac{1}{N}\sum_{j=1}^{N} a_{i,j}^N \psi(x_j^N - x_i^N),$$

where $a_{i,j}^N$ are constant and ψ represents nonlinearity.

- According to the graph limit theory[7], if

$$W^N(s,s_*) = \sum_{i,j=1}^{N} a_{i,j}^N 1_{[\frac{i}{N},\frac{(i+1)}{N})}(s) 1_{[\frac{j}{N},\frac{(j+1)}{N})}(s_*)$$

is uniformly bounded and converges to W, then in the limit $N \to \infty$ we get the non-local diffusive equation,

$$\partial_t x(s,t) = \int_{[0,1]} W(s,s_*)\psi(x(s_*,t) - x(s,t))ds_*.$$

[7] G. S. Medvedev. SIAM J. Math. Anal. 46, 4 (2014), 2743–2766.

Nonlinear subordination

U. Biccari, D. Ko & E. Z., M3AS, 2019, to appear

- $$\dot{x}_i = \frac{1}{N}\sum_{j=1}^{N} a(|x_j - x_i|)(x_j - x_i).$$
- The **Graph limit** model:
$$x_t(s,t) = \int_{[0,1]} a(|x(s_*,t) - x(s,t)|)(x(s_*,t) - x(s,t))ds_*.$$
- The **mean-field limit**:
$$\mu_t(x,t) + \nabla_x(V[\mu]\mu) = 0, \quad \text{where} \quad V[\mu] := \int_X a(x_* - x)\mu(x_*,t)dx_*.$$

Subordination transformation

From non-local "parabolic" to non-local "hyperbolic":
$\mu(x,t) = \int_S \delta(x - x(s,t))ds$.

8

[8] Kinetic / conservation laws: Lions-Perthame-Tadmor, JAMS, 1994

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Motivation: Real life

The number of individuals is small, yet the interaction dynamics and control strategies are complex

The model

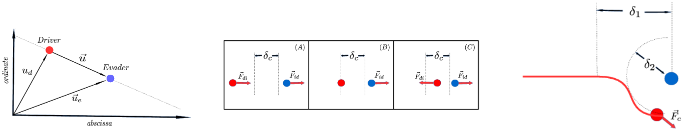
R. Escobedo, A. Ibañez and E.Zuazua, Optimal strategies for driving a mobile agent in a “guidance by repulsion" model, Communications in Nonlinear Science and Numerical Simulation, 39 (2016), 58-72.

We develop and control a **guidance by repulsion** model based on the two-agents framework: *the driver*, which tries to drive the *evader*.

1. The driver follows the evader but cannot be arbitrarily close to it (because of chemical reactions, animal conflict, etc).
2. The evader moves away from the driver but doesn’t try to escape beyond a not so large distance.
3. The driver is faster than the evader.
4. At a critical short distance, the driver can display a **circumvention maneuver** around the evader, forcing it to change the direction of its motion.
5. By adjusting the circumvention maneuver, the evader can be driven towards a desired target or along a given trajectory.

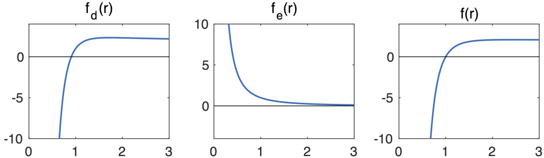
One sheep + one dog + Circumvention control

The control $k(t)$ is chosen in feedback form to align the gate, the sheep and the dog.



In short, the model for $\mathbf{u}_d, \mathbf{u}_e \in \mathbf{R}^2$ can be written as:

$$\begin{cases} \dot{\mathbf{u}}_d = \mathbf{v}_d, \quad \dot{\mathbf{u}}_e = \mathbf{v}_e \\ \dot{\mathbf{v}}_d = -f_d(|\mathbf{u}_d - \mathbf{u}_e|)(\mathbf{u}_d - \mathbf{u}_e) + \kappa(t) g_d(|\mathbf{u}_d - \mathbf{u}_e|)(\mathbf{u}_d - \mathbf{u}_e)^{\perp} - \nu_d \mathbf{v}_d \\ \dot{\mathbf{v}}_e = -f_e(|\mathbf{u}_e - \mathbf{u}_d|)(\mathbf{u}_d - \mathbf{u}_e) - \nu_e \mathbf{v}_e \\ \mathbf{u}_d(0) = \mathbf{u}_d^0, \ \mathbf{u}_e(0) = \mathbf{u}_e^0, \ \mathbf{v}_d(0) = 0, \ \mathbf{v}_e(0) = 0 \end{cases} \tag{1}$$



Symmetric dissipation

When

$$\nu_e/m_e = \nu_d/m_d =: \nu > 0,$$

the model[9] reduces to the dynamics of the relative position, $\mathbf{u} = \mathbf{u}_d - \mathbf{u}_e$,

$$\ddot{\mathbf{u}} + f(|\mathbf{u}|)\mathbf{u} + \nu\dot{\mathbf{u}} = \kappa(t)\mathbf{u}^{\perp}.$$

For the interaction coefficient $f(r)$, we assume

$$f(r) = \begin{cases} \geq 0 & \text{for} \quad r \geq r_c, \\ < 0 & \text{for} \quad 0 < r < r_c \end{cases} \qquad \text{with} \quad f'(r_c) > 0$$

- The equation on the left-hand side follows the motion of damped oscillator under a central potential $\int rf(r)dr$.
- The negativity/positivity of f makes the relative distance $\mathbf{u} \sim r_c$.

Two main regimes arise: Pursuit $\kappa(t) = 0$ / Circumvention $\kappa(t) \neq 0$.

[9] Without loss of generality we assume that $g_d \equiv 1$.

Steady states

For each mode, we have the following steady states which characterize the dynamics:

- Pursuit mode: $\kappa(t) \equiv 0$

$$\mathbf{u}(t) = \mathbf{u}_* \in \mathbb{R}^2 \quad \text{and} \quad \mathbf{v}(t) = (0,0) \quad \text{with} \quad |\mathbf{u}_*| = r_c,$$

where the driver and evader behave uniform linear motions,

$$\mathbf{u}_\ell(t) = -\frac{f_d(\mathbf{u}_*)\mathbf{u}_*}{\nu} t + \mathbf{u}_\ell(0), \quad \ell = d, e.$$

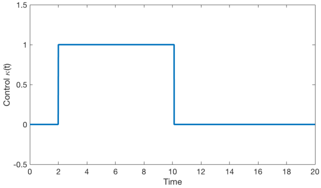
- Circumvention mode, $\kappa(t) \equiv \kappa$

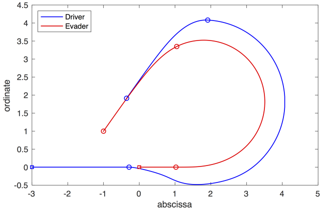
$$\mathbf{u}(t) = r_p \left(\cos\left(\frac{\kappa}{\nu} t\right), \sin\left(\frac{\kappa}{\nu} t\right) \right),$$

where the driver and evader rotates on each circle,

$$\mathbf{u}_\ell(t) = r_\ell \left(\cos\left(\frac{\kappa}{\nu} t + \phi_\ell\right), \sin\left(\frac{\kappa}{\nu} t + \phi_\ell\right) \right) \quad \ell = d, e.$$

Off-Bang-Off control of the evader





Theorem

Let $f(r)$ be as before. Then, for a given destination $\mathbf{u}_f \in \mathbb{R}^2$ and $\mathbf{u}_0 \neq (0,0)$, there exist t_1, t_2, t_f and κ and

$$\kappa(t) = \begin{cases} \kappa & \text{if } \ t \in [t_1, t_2], \\ 0 & \text{if } \ t \in [0, t_1) \cup (t_2, t_f], \end{cases} \qquad \text{such that} \quad \mathbf{u}_e(t_f) = \mathbf{u}_f.$$

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Stability for the linear system

Controlling the system needs a good understanding of the dynamics, especially the asymptotic stability of steady states.

$$\ddot{\mathbf{u}} + \mathbf{u} + \nu\dot{\mathbf{u}} = \kappa\mathbf{u}^{\perp}, \quad \mathbf{u} \in \mathbf{R}^2,$$

which is the damped harmonic oscillator with an additional perpendicular (circumvention) interaction. We want to prove that $\mathbf{u}$ decays to $(0, 0)$.

The standard energy

$$E(t) := \frac{1}{2}(|\mathbf{u}|^2 + |\mathbf{v}|^2)$$

is no more non-increasing from the perpendicular term $\kappa\mathbf{u}^{\perp}$.

$$\dot{E}(t) = -\nu|\mathbf{v}|^2 + \kappa(t)\mathbf{u}^{\perp} \cdot \mathbf{v}.$$

However, we may construct a perturbed energy,

$$F(t) = E(t) + \frac{\nu}{2}\left(\frac{\nu}{2}|\mathbf{u}|^2 + \mathbf{u}\cdot\dot{\mathbf{u}}\right),$$

which fulfills

$$\begin{aligned}\frac{d}{dt}[F(t)] &= -\frac{\nu}{2}|\dot{\mathbf{u}}|^2 - \frac{\nu}{2}|\mathbf{u}|^2 + \kappa(\mathbf{u}^\perp\cdot\dot{\mathbf{u}}) \\ &\le -\frac{1}{2}(\nu-\kappa)(|\mathbf{u}|^2 + |\dot{\mathbf{u}}|^2) = -(\nu-\kappa)E(t).\end{aligned}$$

$\mathbf{u}(t)$ decays exponentially if $\kappa \in (-\nu, \nu)$.

When $\kappa = \nu$, $\mathbf{u}(t) = a(\cos t, \sin t)$, is a periodic solution.

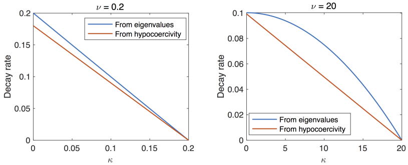

Figure: The decay rate of $E(t)$ from spectral analysis and its estimation from hypocoercivity when $\nu = 0.2$ (left) and $\nu = 20$ (right)

The observed dynamics

From the relative position $\mathbf{u}$, we can recover partial information for the positions $\mathbf{u}_d$ and $\mathbf{u}_e$.

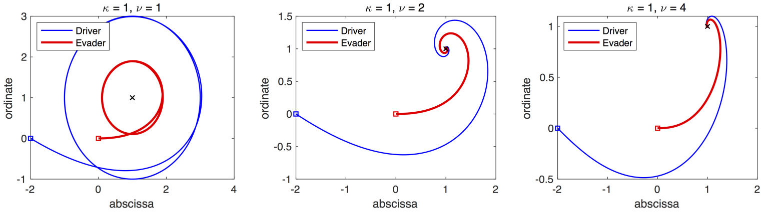

Figure: The trajectory of the driver and evader with $\kappa = 1$ and various ν: $\nu = 1$ (left), 2 (middle), and 3 (right).

This analysis can be used for our nonlinear guidance-repulsion model in order to conclude the asymptotic stability of steady states. Then, we may prove the controllability of the system using the steady states.

Computational feedback control of a sheep-flock by the action of a dog, guided by the shepherd

The feedback law is chosen so to orient the dog and the center of the sheep-flock with the destination gate

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If the driver tracks the evader based on the maximum distance,

$$D(\mathbf{u}) = \max_j |\mathbf{u}_j - \bar{\mathbf{u}}|, \quad \bar{\mathbf{u}} = \frac{1}{N}\sum_{i=1} \mathbf{u}_i,$$

then we conclude similar behaviors as one fat evader.

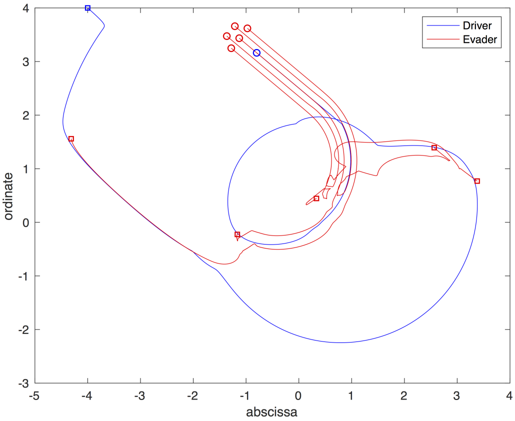

Figure: An example on trajectories of five evaders with a bang-bang control

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Conclusions

- Complex behaviour of networks from a dynamical and control perspective: $N \to +\infty$.
- Optimal location of sensors and actuators for networks is a challenging problem [10]
- $\nu_d \neq \nu_e$?
- Multi-driver modelling and control is challenging.
- Practical applications, with a limited number of individuals, leads to challenging nonlinear dynamical systems.

Make all these analytical and computational developments to be of real use in Social and Behavioral Sciences.

[10] Y Privat, E Trélat & E. Z. ARMA, 2015.