Decay of Partially Dissipative Hyperbolic Systems[1]

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[1]based on K. Beauchard and E. Z, Sharp large time asymptotics for partially dissipative hyperbolic systems, Arch. Rational Mech. Anal., 199 (2011) 177 – 227

Introduction

Partially dissipative linear hyperbolic systems may develop a complex asymptotic behavior as $t \to \infty$ depending on the space dimension, the dimension of the system, and the interaction between the free dynamics and the damping operator.

- The understanding of this issue is relevant, in particular, for analyzing global existence of solutions for nonlinear problems, near constant equilibria.
- The issue turns out to be closely linked to the classical (Kalman/LaSalle) rank condition for the control of finite-dimensional linear systems, the (SK) condition and other notions such as hypoellipticity and hypocoercivity of partially diffusive PDEs.

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The Kalman/LaSalle rank condition

Controllability of finite dimensional linear systems

Let $n, m \in \mathbb{N}^*$ and $T > 0$. Consider the following finite dimensional system:

$$\begin{cases} x'(t) = Ax(t) + Bu(t), & t \in (0, T), \\ x(0) = x^0. \end{cases} \tag{1}$$

Here A is a real $n \times n$ matrix, B is a real $n \times m$ matrix, $x : [0, T] \longrightarrow \mathbb{R}^n$ represents the *state* and $u : [0, T] \longrightarrow \mathbb{R}^m$ the *control*.

System (1) is **controllable** in time $T > 0$ if given any initial and final one $x^0, x^1 \in \mathbb{R}^n$ there exists $u \in L^2(0, T, \mathbb{R}^m)$ such that the solution of (1) satisfies $x(T) = x^1$.

Obviously, in practice $m \leq n$. Of particular interest is the case where m is as small as possible.

The control property does not only depend on the dimensions m and n but on how the matrices A and B interact

$$x'(t) = Ax(t) + Bu(t)$$

Theorem

The system (A, B) is controllable in some time T if and only if

$$\text{rank}\,[B, AB, \cdots, A^{n-1}B] = n. \tag{2}$$

Consequently, if system (1) is controllable in some time $T > 0$ it is controllable in any time.

Note that, the so-called controllability matrix $[B, AB, \cdots, A^{n-1}B]$ is of dimension $n \times nm$. Thus, in the limit case where $m = 1$ (one single control), it is a $n \times n$ matrix. In this case the Kalman rank condition requires it to be invertible.

Key idea of the proof.

$$x(t) = e^{At}x^0 + \int_0^t e^{A(t-s)}Bu(s)ds$$

and

$$e^{AT} = \sum_{k\geq 0} A^k t^k / k!.$$

Thus

$$x(t) = e^{At}x^0 + \int_0^t e^{A(t-s)}Bu(s)ds = e^{At}x^0 + \sum_{k\geq 0} \frac{A^k B}{k!} \int_0^t (t-s)^k u(s)ds$$

From Cayley-Hamilton's Theorem all powers A^k of A with $k \geq n$ are linear combinations of A^k with $k = 0, ..., n-1$. Thus, it is the range of $[B, AB, A^2B, ..., A^{n-1}B]$ that determines the controllability of the system.

Stabilization of finite dimensional linear systems

The controls we have obtained so far are the so called open loop controls. In practice, it is interesting to get closed loop or feedback controls, so that its value is related in real time with the state itself.

Assume, to fix ideas, that A is a skew-adjoint matrix, i. e. $A^* = -A$. In this case, $< Ax, x >= 0$. Consider the system

$$\begin{cases} x' = Ax + Bu \\ x(0) = x^0. \end{cases} \tag{3}$$

When $u \equiv 0$, the energy of the solution of (3) is conserved. Indeed, by multiplying (3) by x, if $u \equiv 0$, one obtains

$$\frac{d}{dt} |x(t)|^2 = 0.$$

Hence,

$$|x(t)| = |x^0|, \quad \forall t \geq 0.$$

We then look for a matrix L such that the solution of system (3) with the *feedback* control law

$$u(t) = Lx(t)$$

has **a uniform exponential decay**, i.e. there exist $c > 0$ and $\omega > 0$ such that

$$|x(t)| \leq ce^{-\omega t}|x^0|$$

for any solution.

In other words, we are looking for matrices L such that the solution of the system

$$x' = (A + BL)x = Dx$$

has an uniform exponential decay rate.

Theorem

If A is skew-adjoint and the pair (A, B) is controllable then $L = -B^*$ stabilizes the system, i.e. the solution of

$$\begin{cases} x' = Ax - BB^*x \\ x(0) = x^0 \end{cases} \tag{4}$$

has an uniform exponential decay.

Proof: With $L = -B^*$ we obtain that

$$\frac{1}{2}\frac{d}{dt}|x(t)|^2 = - < BB^*x(t), x(t) >= - \mid B^*x(t) \mid^2 \leq 0.$$

Hence, the norm of the solution decreases in time.
In fact the decay rate is exponential since we are in finite-dimensions (and all norms are equivalent) and the following unique-continuation property holds: $B^*x \equiv 0 \implies x \equiv 0$.

An example: The harmonic oscillator

Consider the damped harmonic oscillator:

$$mx'' + Rx + kx' = 0, \tag{5}$$

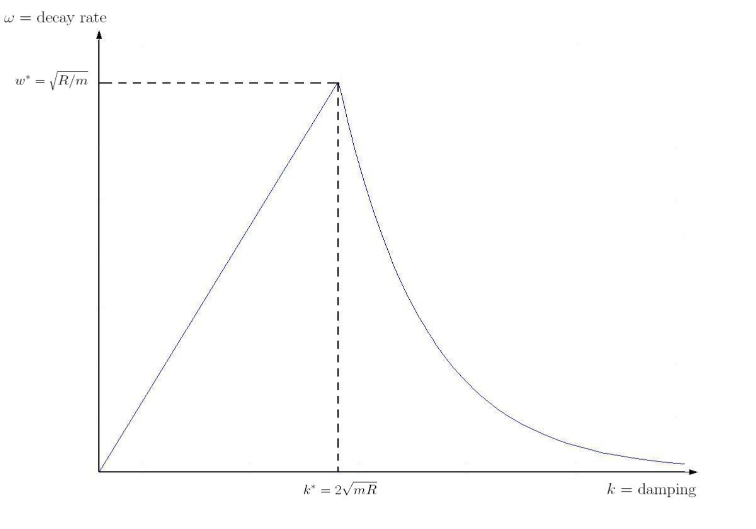
where m, k and R are positive constants.
It is easy to see that the solutions of this equation have an exponential decay property. Indeed, it is sufficient to remark that the two characteristic roots have negative real part. Indeed,

$$mr^2 + R + kr = 0 \Leftrightarrow r_\pm = \frac{-k \pm \sqrt{k^2 - 4mR}}{2m}$$

and therefore

$$\mathrm{Re}\, r_\pm = \begin{cases} -\frac{k}{2m} & \text{if } k^2 \leq 4mR \\ -\frac{k}{2m} \pm \sqrt{\frac{k^2}{4m} - \frac{R}{2m}} & \text{if } k^2 \geq 4mR. \end{cases}$$

We observe here the classical overdamping phenomenon.
Contradicting a first intuition, it is not true that the decay rate increases when the value of the damping parameter k increases.



Systems of balance laws

We consider nonlinear hyperbolic systems of the form

$$\frac{\partial w}{\partial t} + \sum_{j=1}^{m} \frac{\partial F_j(w)}{\partial x_j} = Q(w), \quad x \in \mathbb{R}^m,\ t > 0,$$

$$\begin{array}{rccccc} w: & \mathbb{R} & \times & \mathbb{R}^m & \to & \mathbb{R}^n \\ & (t & , & x) & \mapsto & w(t,x) \end{array}$$

arising in so many applications: fluid mechanics, gas dynamics, traffic flow, ralaxation, ...

Local (in time) smooth solutions are known to exist (C. Dafermos, L. Hsiao-T.-P. Liu, A. Majda, D. Serre,...)
But possible singularities (i.e. shock waves) may arise in finite time.
The nonlinear term Q may play the role of a partial dissipation and help to the existence of global smooth solutions.

Global smooth solutions in a neighborhood of a constant equilibrium: $Q(w^*) = 0$.

THE METHOD = LINEARIZATION + FIXED POINT.

If the linearized dynamics exhibits solutions that decay as $t \to \infty$, a perturbation argument may hopefully allow showing that, locally around the constant equilibrium, solutions are global and decay as well.

One has to distinguish:

- Total dissipation : $Q(w) = -Bw$, $B > 0$
- Partial dissipation: $Q(w) = \begin{pmatrix} 0 & 0 \\ 0 & -D \end{pmatrix} w.$

  Y. Shizuta & S. Kawashima (85), Y. Zeng (99), W. A. Yong (04), S. Bianchini, B. Hanouzet & R. Natalini (03, $m = 1$),...

Example : Isentropic Euler equations with damping

$$\frac{\partial u}{\partial t} - \frac{\partial v}{\partial x} = 0, \quad \frac{\partial v}{\partial t} + \frac{\partial f(u)}{\partial x} = -v$$

Partially dissipative linear hyperbolic systems

$$\frac{\partial w}{\partial t} + \sum_{j=1}^{m} A_j \frac{\partial w}{\partial x_j} = -Bw \ , \ x \in \mathbb{R}^m \ , \ w \in \mathbb{R}^n \qquad (6)$$

$$\begin{array}{c} A_1, \ldots, A_m \\ \text{symmetric} \end{array} \quad \Bigg| \quad B = \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix} \begin{array}{l} \updownarrow n_1 \\ \updownarrow n_2 \end{array} \qquad \begin{array}{c} X^t D X > 0 \\ \forall X \in \mathbb{R}^{n_2} - \{0\} \end{array}$$

Goal: Understand the asymptotic behavior as $t \to \infty$.

Apply Fourier transform:

$$\frac{\partial \hat{w}}{\partial t} = (-B - iA(\xi))\hat{w} \qquad \text{where} \qquad A(\xi) := \sum_{j=1}^{m} \xi_j A_j$$

Lack coercivity :

$\langle [B + iA(\xi)]X, X \rangle = \langle BX, X \rangle = \langle DX_2, X_2 \rangle \not\geqslant c|X|^2$

But possible decay depending on ξ:

$$\exp[(-B - iA(\xi))t] \leqslant Ce^{-\lambda(\xi)t}$$

PARTIALLY DISSIPATIVE LINEAR HYPERBOLIC SYSTEM

$\equiv$

m-PARAMETER (ξ) FAMILY OF FINITE-DIMENSIONAL PARTIALLY DISSIPATIVE n-DIMENSIONAL SYSTEMS.

The asymptotic behavior of solutions is determined by the behavior of the function $\xi \to \lambda(\xi)$ giving the decay rate as a function of ξ.

Example: The dissipative wave equation

$$u_{tt} - u_{xx} + u_t = 0.$$

$$u_t = v_x - u; \quad v_t = u_x.$$

Solutions may be decomposed as

- A high frequency component tending to zero exponentially fast as $t \to 0$;
- A low frequency component with the same decay rate as the heat kernel $t^{-1/2}$ decay in L^∞ for L^1 data.

This corresponds to a function $\lambda(\xi) \sim \min(|\xi|^2, 1)$.

(SK) for partially dissipative linear hyperbolic systems

The pioneering and well-known result by Shizuta-Kawashima (85) may be viewed as a generalization of the example above on the dissipative wave equation.

Under the condition (SK) below
(SK) : $\forall \xi \in \mathbb{R}^m$, **Ker**$(B) \cap \{$**eigenvectors of** $A(\xi)\} = \{0\}$

$$\Rightarrow \exp[(-B - iA(\xi))t] \leqslant Ce^{-\lambda(\xi)t}, \qquad \lambda(\xi) = c\min(1, |\xi|^2)$$

Their proof is based on a (long) linear algebra computation.

Consequences :
(1) $\forall w^0 \in L^1 \cap L^2(\mathbb{R}^m, \mathbb{R}^n), \qquad w = w_l + w_h$

$$\left| w_h(t) \right|_{L^2(\mathbb{R}^m,\mathbb{R}^n)} \leqslant Ce^{-\gamma t} \left| w^0 \right|_{L^2(\mathbb{R}^m,\mathbb{R}^n)}$$
$$\left| w_l(t) \right|_{L^\infty(\mathbb{R}^m,\mathbb{R}^n)} \leqslant Ct^{-\frac{m}{2}} \left| w^0 \right|_{L^1(\mathbb{R}^m,\mathbb{R}^n)}$$

(2) Global smooth solutions for the NL system for initial data close to a constant equilibrium.

Our contributions (K. Beauchard & E. Z., ARMA, 2011)

1st step :
(SK) and rank condition

2nd step :
Measure of the decay rate for $B + iA(\xi)$

3rd step :
Classification of the asymptotic behavior for linear hyperbolic systems (with/without (SK))

4th step :
Nonlinear systems of balance laws : global smooth solutions without (SK)

(SK) and the rank condition

(SK) and rank conditions

Lemma

The (SK) condition is equivalent to imposing the rank condition to the pairs $(A(\xi), B)$ for all values of $\xi \in \mathbb{R}^m$.

$$A \text{ symmetric} \;\Big|\; B = \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix} \begin{array}{l} \updownarrow n_1 \\ \updownarrow n_2 \end{array} \qquad \begin{array}{c} X^t D X > 0 \\ \forall X \in \mathbb{R}^{n_2} - \{0\} \end{array}$$

(SK) : $\mathrm{Ker}(B) \cap \{\text{eigenvectors of } A\} = \{0\}$

$\Leftrightarrow$ The pair (A, B) satisfies the rank condition

$\Leftrightarrow$ $Be^{At}X \equiv 0$ for all $t > 0 \implies X \equiv 0$

$\Leftrightarrow$ $\displaystyle\sum_{k=0}^{n-1} |BA^k X|^2$ is a norm on $\mathbb{R}^n$ (the control-quadratic form)

- The (SK) condition requires the conditions above to be satisfied for all $\xi \in \mathbb{R}^m$.
- In $1-d$ ($m=1$) the (SK) condition is sharp. Whenever it fails, travelling wave solutions with L^2-profiles exist, thus making the decay of solutions impossible. This is so because, in $1-d$, the rank condition holds for all $\xi \in \mathbb{R}$ if and only if the pair (A, B) sastifies the rank condition.
- This is not true in the muti-dimensional case since $A(\xi) = A(\xi) := \sum_{j=1}^{m} \xi_j A_j$ depends on ξ in a non-trivial way.
- In view of the analysis above it is more natural to analyse the positive definiteness of the quadratic form $\sum_{k=0}^{n-1} |BA(\xi)^k X|^2$, as function of ξ.
- This will illustrate the existence of many other scenarios that (SK) excludes, except in one space dimension ($m=1$). In the multi-dimensional case (SK) is a sufficient condition for decay, but is far from being necessary.

Decay rate for $B + iA(\xi)$ as a function of ξ

A measure of the decay rate as a function of ξ

$A_1, \dots, A_m$ symmetric | $B = \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix} \begin{matrix} \updownarrow n_1 \\ \updownarrow n_2 \end{matrix}$ | $A(\xi) := \sum_{j=1}^{m} \xi_j A_j$

$\xi = \rho\omega \in \mathbb{R}^m \qquad \rho > 0 \qquad \omega \in S^{m-1} \qquad (m_k) \uparrow$ well chosen

$$N_{*,\epsilon}(\omega) := \min\{\sum_{k=0}^{n-1} \epsilon^{m_k} |BA(\omega)^k x|^2; x \in S^{n-1}\}.$$

Theorem

$\exists \epsilon_* > 0, c > 0$ *such that* $\forall \epsilon \in (0, \epsilon_*)$,

$$\exp[(-B - i\rho A(\omega))t] \leqslant 2e^{-cN_{*,\epsilon}(\omega)\min\{1,\rho^2\}t}.$$

Remark : **(SK)** $\Leftrightarrow N_{*,\epsilon}(\omega) \geqslant N_{*,\epsilon} > 0, \forall \omega \in S^{m-1}$.
In general, $N_{*,\epsilon}(\omega)$ may vanish for some values of $\omega \in S^{m-1}$, in which case the decomposition of solutions and its asymptotic form is more complex.

Our proof:

- Is based on energy arguments and the construction of a suitable Lyapunov functional that allows exhibiting the exponential decay rate. In that sense it is similar to the techniques employed for proving decay rates for dissipative wave equations, and the works by C. Villani et al. on the decay for kinetic equations and hypocoercivity.
- Yields the result by Shizuta-Kawashima in a simpler way, but shows that it only covers one of the many possible behaviors one may encounter for $m \geq 2$.
- Provides quantitative estimates on the decay rate as a function of ξ that can be used for a better understanding of the nonlinear problem.

An example: the dissipative wave equation.

$$u_{tt} - u_{xx} + u_t = 0.$$

$$w_{tt} + \xi^2 w + w_t = 0.$$

$$e_\xi(t) = \frac{1}{2}[|w_t|^2 + \xi^2 |w|^2].$$

$$\frac{de_\xi(t)}{dt} = -|w_t|^2.$$

The exponential decay rate does not come directly out of this. But it is easy to check that

$$f_\xi(t) = e_\xi(t) + \varepsilon w w_t,$$

is, for ε small enough, such that

$$f_\xi \sim e_\xi$$

and, on the other hand,

$$\frac{df_\xi(t)}{dt} \leq -c(\xi, \varepsilon) f_\xi(t).$$

Indeed,

$$\frac{df_\xi(t)}{dt} = -|w_t|^2 + \varepsilon|w_t|^2 + \varepsilon w w_{tt} = -(1-\varepsilon)|w_t|^2 - \varepsilon\xi^2 w^2 - \varepsilon w w_t$$

$\leq -(1 - \varepsilon - \frac{\varepsilon}{2\xi^2})|w_t|^2 - \frac{\varepsilon\xi^2}{2} w^2.$

Then, it suffices to take:

$$\varepsilon = \min\left(\xi^2, \frac{1}{4}\right).$$

Note that there is an extensive literature on the extensions of this result to nonlinear problems (M. Nakao, A. Haraux, ...).

Decay rate for $B + iA(\xi)$: $\rho < 1$

$$\omega \in S^{m-1}: \sum_{k=0}^{n-1} \epsilon^{m_k} |BA(\omega)^k x|^2 \geq N_{*,\epsilon} > 0, \forall x \in S^{n-1}.$$

$$\dot{x} = (-B - i\rho A(\omega))x, \qquad \textbf{Goal :} |x(t)| \leqslant 2|x_0| e^{-cN_{*,\epsilon}\rho^2 t}$$

Strategy : find $\mathcal{L}(x) \sim |x|^2$ such that $\frac{d\mathcal{L}}{dt} \leqslant -c\rho^2 N_{*,\epsilon}\mathcal{L}$

$$\mathcal{L}(x) = |x|^2 + \rho \sum_{k=1}^{n-1} \epsilon^{m_k} \mathrm{Im}\left(\langle A(\omega)^k BBA(\omega)^{k-1}x, x\rangle\right)$$

$$\begin{aligned}
\frac{d\mathcal{L}}{dt} &= -2\mathrm{Re}\left(\langle (B + i\rho A_\omega)x, x\rangle\right) \\
&\quad -\rho \textstyle\sum_{k=1}^{n-1} \epsilon^{m_k} \mathrm{Im}\left(\langle (A_\omega^t)^k B^t BA_\omega^{k-1}(B + i\rho A_\omega)x, x\rangle\right) \\
&\quad -\rho \textstyle\sum_{k=1}^{n-1} \epsilon^{m_k} \mathrm{Im}\left(\langle (A_\omega^t)^k B^t BA_\omega^{k-1}x, (B + i\rho A_\omega)x\rangle\right) \\
&\leqslant -2C_1|Bx|^2 - \rho^2 \textstyle\sum_{k=1}^{n-1} \epsilon^{m_k} |BA_\omega^k x|^2 \\
&\quad +\rho \textstyle\sum_{k=1}^{n-1} \epsilon^{m_k} |Bx| \left[\left| BA_\omega^{k-1} \right| |BA_\omega^k x| + \left| BA_\omega^k \right| |BA_\omega^{k-1}x| \right] \\
&\quad +\rho^2 \textstyle\sum_{k=1}^{n-1} \epsilon^{m_k} |BA_\omega^{k-1}x| |BA_\omega^{k+1}x|
\end{aligned}$$

Asymptotic behavior for linear hyperbolic systems (with or without (SK))

The set of degeneracy

The minimum of the control quadratic form
$N_{*,\epsilon}(\omega) := \min\{\sum_{k=0}^{n-1} \epsilon^{m_k} |BA(\omega)^k x|^2 ; x \in S^{n-1}\}$
measures the decay rate for $B + i\rho A(\omega)$

$N_{*,\epsilon}(\omega) > 0 \;\Leftrightarrow\; \mathrm{Ker}(B) \cap \{\text{eigenvectors of } A(\omega)\} = \{0\}$
$\Leftrightarrow$ $(A(\omega), B)$ satisfies the rank condition.

The set of degeneracy :
$\mathcal{D}(B + iA(\xi)) = \{\xi \in \mathbb{R}^m ; rank[B|BA(\xi)|...|BA(\xi)^{n-1}] < n\}$
is an algebraic submanifold

$\rightarrow$ either $|\mathcal{D}| = 0 \Leftrightarrow N_{*,\epsilon} > 0$ a.e. $\Rightarrow$ strong L^2 stability
$\rightarrow$ or $\mathcal{D} = \mathbb{R}^m$: $\exists$ non dissipated solutions
Decomposition ?

$|\mathcal{D}| = 0$, $n_1 = 1$ (B is effective in all but one components)

Theorem

When $n_1 = 1$, $\mathcal{D}$ *is a vector subspace of* $\mathbb{R}^m$ *and*

$$N_{*,\epsilon}(\omega) \geqslant c \min\{1, dist(\omega, \mathcal{D})^2\}, \forall \omega \in S^{m-1}.$$

As a consequence we have the following new decomposition

$$w = w_h + w_l + w_{new}$$

with

$$\left|w_h(t)\right|_{L^2(\mathbb{R}^m,\mathbb{R}^n)} \leqslant Ce^{-t}\left|w^0\right|_{L^2(\mathbb{R}^m,\mathbb{R}^n)}$$
$$\left|w_l(t)\right|_{L^\infty(\mathbb{R}^m,\mathbb{R}^n)} \leqslant Ct^{-\frac{m}{2}}\left|w^0\right|_{L^1(\mathbb{R}^m,\mathbb{R}^n)}$$
$$\left|w_{new}(t)\right|_{L^\infty(\mathbb{R}^m,\mathbb{R}^n)} \leqslant Ct^{-\frac{1}{2}}\left|w^0\right|_{*}$$

Example: $n = m = 2$; $\mathcal{D} = \{(\xi_1, \xi_2) : a^1_{21}\xi_1 + a^2_{21}\xi_2 = 0\}$.

Classification of the asymptotic behaviors

$$\frac{\partial w}{\partial t} + \sum_{j=1}^{m} A_j \frac{\partial w}{\partial x_j} = -Bw, \quad x \in \mathbb{R}^m, \quad w \in \mathbb{R}^n$$

	(SK)	$\mathcal{D}$	L^2 stability	decomposition
$m=1$	(SK)	$\{0\}$	yes	$e^{-t} + \frac{1}{\sqrt{t}}$
$\forall n$	no (SK)	$\mathbb{R}$	no	$e^{-t} + \frac{1}{\sqrt{t}}$
				$+$trav. waves
$n=2$	(SK)	$\{0\}$	yes	$e^{-t} + \frac{1}{t}$
$\forall m$	no (SK)	hyperplane	yes	$e^{-t} + \frac{1}{t} + \frac{1}{\sqrt{t}}$
	no (SK)	$\mathbb{R}^m$	no	$e^{-t} +$ trav. wave
$\forall n$	(SK)	$\{0\}$	yes	$e^{-t} + \frac{1}{t^{m/2}}$
$\forall m$	no $\& n_1 = 1$	vect. subsp.	yes	$e^{-t} + \frac{1}{t^{m/2}} + \frac{1}{\sqrt{t}}$
	no (SK)	submanifold	yes	*open*
		$\mathbb{R}^m$	no	*open*

Existence of global smooth solutions with (SK) (Yong, 04)

$$\frac{\partial w}{\partial t} + \sum_{j=1}^{m} \frac{\partial F_j(w)}{\partial x_j} = -Bw \qquad B := \begin{array}{cc} \begin{pmatrix} 0 & 0 \\ 0 & I_{n_2} \end{pmatrix} & \begin{array}{c} \updownarrow n_1 \\ \updownarrow n_2 \end{array} \end{array}$$

Theorem

Assume W_e constant equilibrium with (SK) on its linearized system.
Let $s \geqslant [m/2] + 2$ be an integer. There exists $\delta > 0$ such that, $\forall w_0 \in W_e + H^s(\mathbb{R}^m, \mathbb{R}^n)$ with $\left| w_0 - W_e \right|_{H^s} \leqslant \delta$ there exists a unique global solution $w \in C^0((0, +\infty)_t, W_e + H^s)$.

The proof uses local existence, a continuation argument and the decay rate of the linearized system around W_e.

A SIMILAR RESULT HOLDS IN THE MORE GENERAL SETTING IN WHICH WE HAVE ENLARGED THE DECOMPOSITION RESULTS IN THE ABSENCE OF (SK).

Connections with hypoellipticity & hypocoercivity

Hypoellipticity $\equiv$ The fundamental solution is C^∞ away form the diagonal (....L. Hörmander, 1968....)

$$\partial_t u - \sum_{j,k=1}^{n} a_{jk}\partial^2_{jk} u + \sum_{j,k=1}^{n} b_{jk} x_j \partial_k u = 0.$$

Example: Kolmogorov equation: $u_t - u_{xx} - xu_y = 0$.
After applying Fourier transform:

$$U_t - A(\xi,\xi)U - \sum_{j,k=1}^{n} b_{jk}\partial_j(\xi_k U) = 0.$$

Hypoellipticity fails iff $A(e^{Bs\xi}, e^{Bs\xi}) = 0$ for all $s >$ for some $\xi \neq 0$.

Hypocoercivity (....C. Villani, 2006,....)

$$f_t + Lf = 0$$

$$L = A^* A + B, \qquad \text{where } B^* = -B.$$

Example: Similar models as above but in the context of kinetic equations: Fokker-Planck equation:

$$f_t - \Delta_v f + v \cdot \nabla_x f - \nabla V(x) \cdot \nabla_v f - \nabla_v \cdot (vf) = 0.$$

Despite the lack of coercivity of L, under suitable assumptions on the commutators of A and B, one can build a modified energy or suitable Lyapunov functional (similar to our construction of the functional $\mathcal{L}$) in which the time exponential decay can be proved.

Conclusion

Control theoretical tools applied to analyze the decay rate for $B + iA(\xi)$ yield:

- For linear hyperbolic systems :
  a (yet non complete) classification for the asymptotic behaviors, with and without (SK),
- for nonlinear systems of conservation laws :
  existence of global smooth solutions around some degenerate constant equilibria.

Future work:

- To make the classification of linear systems complete[2]
- Significantly enlarge the class of nonlinear systems for which global existence holds.

[2] G. Ottaviani and B. Sturmfels, Matrices with eigenvectors in a given subspace, Proc. Amer. Math. Soc. 141 (2013), 1219-1232.

Existence of global smooth solutions : proof

$$\frac{\partial w}{\partial t} + \sum_{j=1}^{m} A_j(w) \frac{\partial w}{\partial x_j} = -Bw$$

Strategy : local existence + continuation argument

$$\left| w(T) - W_e \right|_{H^s}^2 + \int_0^T \left| w_2 \right|_{H^s}^2 + C_1 N_*(W_e^p)^2 \left| \nabla_x w \right|_{H^{s-1}}^2 \leqslant \left| w_0 - W_e \right|_{H^s}$$

1st step : L^2-estimate

$$\left| w(T) - W_e \right|_{L^2}^2 + 2 \int_0^T \left| w_2 \right|_{L^2}^2 \leqslant \left| w_0 \right|_{L^2}^2$$

2nd step : H^s-estimate

$$\left| w(T) \right|_{H^s}^2 + 2 \int_0^T \left| w_2 \right|_{H^s}^2 \leqslant \left| w_0 \right|_{H^s}^2 + CN_s(T) \int_0^T \left| \nabla w \right|_{H^{s-1}}^2$$

$$N_s(T) := \sup\{ \left| w(t) - W_e \right|_{H^s}; t \in (0, T) \}$$

3rd step : estimate on $\int_0^T \left|\nabla w\right|^2_{L^2}$

$$\frac{\partial w}{\partial t} + \sum_{j=1}^{m} A_j(W_e^p)\frac{\partial w}{\partial x_j} = -Bw + h_p$$

$N_*(W_e^p)$= hypocoercivity for the linear system

$$\begin{aligned}&\left|w(T) - W_e\right|^2_{H^s} + \int_0^T \left|w_2\right|^2_{H^s} + N_*(W_e^p)^2\left|\nabla w\right|^2_{H^{s-1}}\\ \leqslant &\left|w_0 - W_e\right|^2_{H^s} + (N_s(T) + |W_e - W_e^p|^2)\int_0^T \left|\nabla_x w\right|^2_{H^{s-1}}\end{aligned}$$

If $N_s(T) + |W_e - W_e^p|^2) < N_*(W_e^p)^2$ then
$\left|w(T) - W_e\right|_{H^s} < \left|w_0 - W_e\right|_{H^s}$.

This always holds when

$$\left|w_0 - W_e\right|_{H^s} < \tfrac{1}{2}N_*(W_e^p)^2 \qquad\qquad \left|W_e - W_e^p\right| < \tfrac{1}{2}N_*(W_e^p)$$