Open problems in PDE control

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Elliptic optimal control

Consider the elliptic problem, where $u = u(x)$ is the control and $y = y(x)$ the state:

$$\begin{cases} -\Delta y + y^3 = u(x) & \text{in } \Omega \\ y = 0 & \text{on } \partial\Omega. \end{cases} \qquad (1)$$

For all $u \in L^2(\Omega)$ there is an unique solution $y \in H_0^1(\Omega) \cap L^4(\Omega)$:

$$\min_{H_0^1(\Omega)\cap L^4(\Omega)} \frac{1}{2}\int_\Omega |\nabla y|^2 dx + \frac{1}{4}\int_\Omega |y|^4 dx - \int_\Omega uy dx.$$

Consider the optimal control problem:

$$\min_{u\in L^2(\Omega)} J(u) = \frac{1}{2}|y - z|^2_{L^2(\Omega)} + \frac{1}{2}|u|^2_{L^2(\Omega)}. \qquad (2)$$

The minimum exists[1]

OP1: Is the global minimiser unique? Is it a strict minimum?

[1] A consequence of the straightforward application of the Direct Method of the Calculus of Variations

A bit easier

Consider the finite-dimensional problem

$$Ay + ||y||^2 y = u, \tag{3}$$

A symmetric and $A \geq \alpha > 0$. For all $u \in \mathbf{R}^N$ there is an unique $y \in \mathbf{R}^N$:

$$\min_{y \in \mathbf{R}^N} \frac{1}{2} < Ay, y > + \frac{1}{4} ||y||^4 - < u, y > .$$

Consider the optimal control problem:

$$\min_{u \in \mathbf{R}^N} \; J(u) = \frac{1}{2} ||y - z||^2 + \frac{1}{2} ||u||^2. \tag{4}$$

The minimum exists.

OP0: Is the global minimiser unique? Is it a strict minimum? [2]

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[2] In some very particular cases Dario Pighin (PhD student in our team) has shown that the global minimiser is not unique

Much easier!

Consider 1-d problem

$$y + |y|^2 y = u. \tag{5}$$

For all $u \in \mathbf{R}$ there is an unique solution $y \in \mathbf{R}$:

$$\min_{y \in \mathbf{R}} \frac{1}{2} y^2 + \frac{1}{4} y^4 - uy.$$

Consider the optimal control problem:

$$\min_{u \in \mathbf{R}} \; J(u) = \frac{1}{2}|y - z|^2 + \frac{1}{2} u^2. \tag{6}$$

The minimum exists.

Replace $u = y + |y|^2 y$ in the functional J. Then, J, written as a function of y is strictly convex and the minimiser unique:

$$J(u) = \frac{1}{2}|y - z|^2 + \frac{1}{2}|y + |y|^2 y|^2 = \frac{1}{2}|y - z|^2 + \frac{1}{2}\left[y^2 + y^6 + 2y^4\right].$$

Back to the PDE: The answer is YES when the target z is small enough.

If $\bar{u}$ is a minimiser

$$J(\bar{u}) = \frac{1}{2}|\bar{y} - z|^2_{L^2(\Omega)} + \frac{1}{2}|\bar{u}|^2_{L^2(\Omega)} \leq J(0) = \frac{1}{2}|z|^2_{L^2(\Omega)}$$

and, in particular,

$$|\bar{u}|_{L^2(\Omega)} \leq |z|_{L^2(\Omega)}.$$

If z is small, $\bar{u}$ is small too, and so is the solution $\bar{y}$. Then $(\bar{y})^3 \sim 0$ and the elliptic PDE under consideration is nearly linear, the functional J nearly quadratic and therefore:

1. The minimiser $\bar{u}$ is small and unique;
2. It is a strict minimum: $D^2J(\bar{u}) > 0$.

What happens when z is large?

The optimality system does not seem to help much

We could analyse the minimiser as a solution of the system of Optimality (OS), involving the state y and the adjoint or costate p, the vector (y, p). But the system under consideration doers not seem to have any monotocity property allowing to conclude the uniqueness:

$$\begin{cases} -\Delta y + y^3 + p = 0 & \text{in } \Omega \\ y = 0 & \text{on } \partial\Omega \\ -\Delta p + 3y^2 p - y = -z & \text{in } \Omega \\ p = 0 & \text{on } \partial\Omega. \end{cases} \tag{7}$$

We change variables $u \to y$

Recall that

$$u = -\Delta y + y^3.$$

Then

$$J(u) = G(y) = \frac{1}{2}|y - z|^2_{L^2(\Omega)} + \frac{1}{2}| - \Delta y + y^3|^2_{L^2(\Omega)}$$

$$= \frac{1}{2}\int_\Omega [|y-z|^2 + |-\Delta y + y^3|^2]dx = \frac{1}{2}\int_\Omega [|y-z|^2 + |-\Delta y|^2 + y^6 + 6|\nabla y|^2 y^2]$$

The functional G is now defined in the space $[H^2 \cap H^1_0](\Omega) \cap L^6(\Omega)$ and looks very nice

But the term $\int_\Omega |\nabla y|^2 y^2 dx$ is not necessarily convex.

Parabolic optimal control

Consider the parabolic problem

$$\begin{cases} y_t - \Delta y + y^3 = u(x,t)\chi_\omega & \text{in } \Omega \times (0,T) \\ y = 0 & \text{on } \partial\Omega \times (0,T) \\ y(0) = y^0 & \text{in } \Omega. \end{cases} \tag{8}$$

For all $u = u(x,t) \in L^2(\Omega \times (0,T))$ and $y^0 \in L^2(\Omega)$ there is an unique solution
$y \in C([(0,T]; L^2(\Omega)) \cap L^2(0,T; H_0^1(\Omega)) \cap L^4(\Omega \times (0,T))$.

Consider the optimal control problem:

$$\min_{u \in L^2(\Omega \times (0,T))} J(u) = \frac{1}{2}\int_0^T |y - z|^2_{L^2(\Omega)} dt + \frac{1}{2}\int_0^T |u|^2_{L^2(\Omega)} dt. \tag{9}$$

The minimum exists.

OP2: Is the global minimiser unique? Is it a strict minimum?

Easier...

Same problems in the ODE setting for

$$\begin{cases} y' + Ay + ||y||^2 y = Bu & \text{in } (0, T) \\ y(0) = y^0, \end{cases} \tag{10}$$

with $A \in \mathcal{M}_{N\times N}$ and $B \in \mathcal{M}_{N\times M}$.
For all $u = u(t) \in L^2(0, T; \mathbf{R}^M)$ and $y^0 \in \mathbf{R}^N$ there is an unique solution $y \in C([(0, T]; \mathbf{R}^N)$.

Consider the optimal control problem:

$$\min_{u\in L^2(0,T;\mathbf{R}^M)} J(u) = \frac{1}{2}\int_0^T ||y - z||^2 dt + \frac{1}{2}\int_0^T ||u||^2 dt. \tag{11}$$

The minimum exists.

OP2b: Is the global minimiser unique? Is it a strict minimum?

The turnpike property

Many control problems arising in engineering, biomedicine and social sciences, lead to natural questions of control in long time horizons.

1. Sustainable growth
2. Cronical diseases
3. New generation of supersonic aircrafts

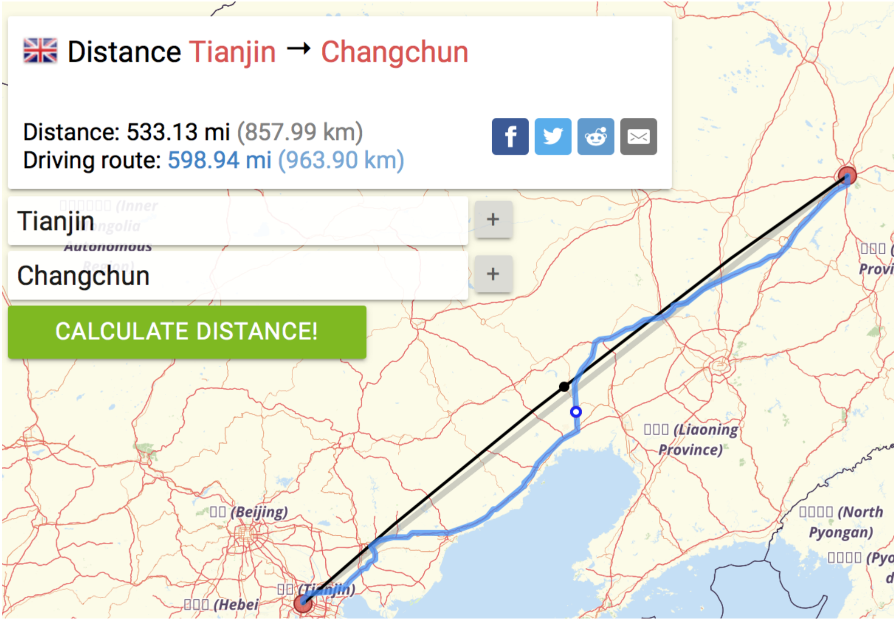
Challenges:

1. Develop specific tools for long time control horizons.
2. Build numerical schemes capable of reproducing accurately the control dynamics in long time intervals (geometric integration, asymptotic preserving schemes).

Origins

Although the idea goes back to John von Neumann in 1945, Lionel W. McKenzie traces the term to Robert Dorfman, Paul Samuelson, and Robert Solow's "Linear Programming and Economics Analysis" in 1958, referring to an American English word for a Highway:

> *... There is a fastest route between any two points; and if the origin and destination are close together and far from the turnpike, the best route may not touch the turnpike. But if the origin and destination are far enough apart, it will always pay to get on to the turnpike and cover distance at the best rate of travel, even if this means adding a little mileage at either end.*


Distance Tianjin → Changchun
Distance: 533.13 mi (857.99 km)
Driving route: 598.94 mi (963.90 km)
Tianjin
Changchun
CALCULATE DISTANCE!
(Liaoning
Province)
(Beijing)
(Tianjin)
(Hebei
(North
Pyongan)

Tunrpike property $\equiv$ Asymptotic simplification

The turnpike property...

1. ... ensures that optimal strategies for the steady-state problem lead to nearly optimal ones for the time-dependent dynamics.
2. ... is employed systematically much beyond the class of problems for which the principle can be proved to hold rigorously.
3. ... can be of use in many contexts such as mesh adaptivity, parameter-dependent problems, etc, to make problems time-independent.

[3] The problem is much simpler in the context of time-independent optimal controls for the time-evolution problem

$$\begin{cases} y_t - \Delta y + y^3 = u(x)\chi_\omega & \text{in } \Omega \times (0, T) \\ y = 0 & \text{on } (0, T) \times \partial\Omega \\ y(0) = y_0 \in L^2 \ , \end{cases} \tag{12}$$

with controls $u = u(x)$ independent of time.

Consider the optimal control problem:

$$\min_{u \in L^2(\omega)} J^T(u) = \frac{1}{2} \int_0^T |y(t) - z|^2_{L^2(\omega_0)} dt + \frac{1}{2} |u|^2_{L^2(\omega)} \ , \tag{13}$$

[3] A. Porretta & E. Z., Remarks on long time versus steady state optimal control, Springer INdAM Series "Mathematical Paradigms of Climate Science", F. Ancona et al. eds, 15, 2016, 67-89.

and the steady state version

$$\begin{cases} -\Delta y + y^3 = u\chi_\omega & \text{in } \Omega \\ y = 0 & \text{on } \partial\Omega\,, \\ \min_{u\in L^2(\omega)} \; J(u) = \frac{1}{2}\left[|u|^2_{L^2(\omega)} + |y-z|^2_{L^2(\omega_0)}\right]\,. \end{cases} \tag{14}$$

Γ-convergence arguments allow showing convergence as $T \to \infty$:

We have

$$u_T \to u \quad \text{in} \quad L^2(\Omega).$$

OP3: Is the convergence rate exponential?

$$||u_T - u||_{L^2(\Omega)} \leq C\exp(-\alpha T)$$

Can this exponential convergence property be seen in the Optimality System?

Time-evolution optimality system

$$
\begin{cases}
y_t - \Delta y + y^3 = -\frac{1}{T}\int_0^T p(x,t)dt & \text{in } \Omega \\
y = 0 & \text{on } \partial\Omega \\
y(x,0) = y^0(x) & \text{in } \Omega \\
-p_t - \Delta p + 3y^2 p = y - z & \text{in } \Omega \\
p = 0 & \text{on } \partial\Omega \\
p(x,T) = 0 & \text{in } \Omega.
\end{cases}
\tag{15}
$$

The steady-state version:

$$
\begin{cases}
-\Delta y + y^3 = -p(x) & \text{in } \Omega \\
y = 0 & \text{on } \partial\Omega \\
-\Delta p + 3y^2 p = y - z & \text{in } \Omega \\
p = 0 & \text{on } \partial\Omega.
\end{cases}
\tag{16}
$$

[4] Consider now the semilinear heat equation:

$$\begin{cases} y_t - \Delta y + y^3 = u(x,t)1_\omega & \text{in} \quad Q \\ y = 0 & \text{on} \quad \Sigma \\ y(x,0) = y^0(x) & \text{in} \quad \Omega. \end{cases} \tag{17}$$

Consider the minimisation problem:

$$\min_f \Big[\frac{1}{2}\int_0^T \int_\Omega |y - y_d|^2 dxdt + \int_0^T \int_\omega u^2 dxdt\Big].$$

[4] Extension to $2d$ NS by S. Zamorano, UChile

The optimality system reads:

$$y_t - \Delta y + y^3 = -p 1_\omega \text{ in } Q$$

$$y = 0 \text{ on } \Sigma$$

$$y(x,0) = y^0(x) \text{ in } \Omega$$

$$-p_t - \Delta p + 3y^2 p = y - y_d \text{ in } Q$$

$$p = 0 \text{ on } \Sigma$$

$$p(x,T) = 0 \text{ in } \Omega.$$

And the linearised optimality system, around the optimal steady solution $(\bar{y}, \bar{p})$ is as follows:

$$\eta_t - \Delta\eta + 3(\bar{y})^2\eta = -q 1_\omega \text{ in } Q$$

$$\eta = 0 \text{ on } \Sigma$$

$$\eta(x, 0) = 0 \text{ in } \Omega$$

$$-q_t - \Delta q + 3(\bar{y})^2 q + 6\bar{y}\bar{p}\eta = \eta \text{ in } Q$$

$$q = 0 \text{ on } \Sigma$$

$$q(x, T) = 0 \text{ in } \Omega.$$

The linearised optimality system reads as follows:

$$\eta_t - \Delta\eta + 3(\bar{y})^2\eta = -q1_\omega, \quad -q_t - \Delta q + 3(\bar{y})^2 q = (1 - 6\bar{y}\bar{p})\eta$$

This is the optimality system for a LQ control problem of the model

$$\eta_t - \Delta\eta + 3(\bar{y})^2\eta = f1_\omega$$

and the cost

$$\min_f \left[\frac{1}{2}\int_0^T \int_\Omega \rho(x)|\eta|^2 dxdt + \int_0^T \int_\omega f^2 dxdt\right],$$

$$\rho(x) = 1 - 6\bar{y}(x)\bar{p}(x).$$

And the turnpike property holds as soon as $\rho(x) \geq \delta > 0$:

$$||y_T(t) - \bar{y}||_{L^2(\Omega)} + ||u_T(t) - \bar{u}||_{L^2(\omega)} \leq C[\exp(-\alpha t) + \exp(-\alpha(T-t))].$$

This holds if $\bar{y}$ and $\bar{p}$ are small enough, and this is automatically implied as soon as the target y_d is small enough. [5]

[5] A. Porretta & E. Z., Long time versus steady state optimal control, SIAM J. Cont. Optim., 51 (6) (2013), 4242-4273.

The second order optimality conditions for the minimiser of the steady-state problem guarantee that the corresponding steady-state functional is semidefinite positive[6]. But this does not imply that the time-dependent functional is positive[7].

OP4.
Can we get rid of the smallness condition on the target?

OP5.
In case there are several minimisers can we determine the basin of attraction of each of the turnpikes?

[6] E. Casas & M. Mateos, EHF2016 Lecture Notes, 2016.

[7] Private communication D. Pighin

Oscillatory nature of optimal parabolic controls

Sharp estimates for heat control

Let $n \geq 1$ and $T > 0$, Ω be a simply connected, bounded domain of $\mathbb{R}^n$ with smooth boundary Γ, $Q = (0, T) \times \Omega$ and $\Sigma = (0, T) \times \Gamma$:

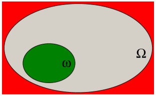
$$\begin{cases} u_t - \Delta u = f 1_\omega & \text{in} \quad Q \\ u = 0 & \text{on} \quad \Sigma \\ u(x, 0) = u^0(x) & \text{in} \quad \Omega. \end{cases} \tag{18}$$

1_ω = the characteristic function of ω of Ω where the control is active.

Assume $u^0 \in L^2(\Omega)$, $f \in L^2(Q)$ so that (22) admits an unique solution

$$u \in C\left([0, T]\,; L^2(\Omega)\right) \cap L^2\left(0, T; H_0^1(\Omega)\right).$$

$u = u(x, t) = solution = state,\ f = f(x, t) = control$



Given u_0 and $T > 0$ find $f = f(x, t)$ such that

Well known result (Fursikov-Imanuvilov, Lebeau-Robbiano,...) :
The system is null-controllable in any time T and from any open non-empty subset ω of Ω.
The control of minimal L^2-norm can be found by minimizing

$$J_0(\varphi^0) = \frac{1}{2}\int_0^T \int_\omega \varphi^2 dxdt + \int_\Omega \varphi(0) u^0 dx \tag{19}$$

over the space of solutions of the adjoint system:

$$\begin{cases} -\varphi_t - \Delta\varphi = 0 & \text{in } Q \\ \varphi = 0 & \text{on } \Sigma \\ \varphi(T,x) = \varphi^0(x) & \text{in } \Omega. \end{cases} \tag{20}$$

Obviously, the functional is continuous and convex from $L^2(\Omega)$ to $\mathbb{R}$ and coercive because of the observability estimate:

$$\| \varphi(0) \|^2_{L^2(\Omega)} \leq C \int_0^T \int_\omega \varphi^2 dxdt, \quad \forall \varphi^0 \in L^2(\Omega). \tag{21}$$

One has in fact

$$\int_0^T \int_\Omega e^{\frac{-A}{(T-t)}} \varphi^2 dxdt \leq C \int_0^T \int_\omega \varphi^2 dxdt.$$

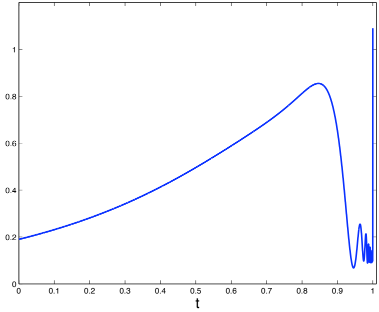
Or, in Fourier series,

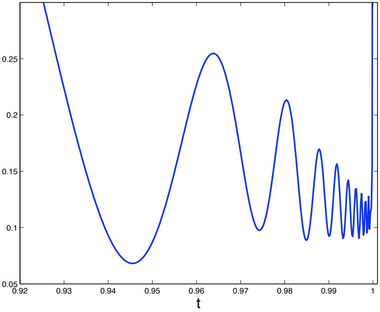
$$\sum_{k\geq 1} \exp(-B\sqrt{\lambda_k})|\hat{\varphi}_k^T|^2 \leq C \int_0^T \int_\omega \varphi^2 dxdt.$$

OP5: Characterize the best constants A and B

Oscillatory nature of controls

Numerical simulations exhibit an oscillatory nature of controls, which is very much compatible with the estimates above, that show that the adjoint state at time T, belong to a very wide space.





OP5: Can this oscillatory pattern be characterised?

Oscillatory nature of controls

OP6: Can one estimate (lower and upper bounds) the gap between consecutive zeroes:

$$\sum_{k\geq 1} a_k \exp(-k^2(T-t))$$

with

$$\sum_{k\geq 1} |a_k|^2 \exp(-ck) < +\infty.$$

Optimal design of waves

Internal stabilization of waves: Let ω be an open subset of Ω. Consider:

$$\begin{cases} y_{tt} - \Delta y = -y_t 1_\omega & \text{in} \quad Q = \Omega \times (0, \infty) \\ y = 0 & \text{on} \quad \Sigma = \Gamma \times (0, \infty) \\ y(x,0) = y^0(x), y_t(x,0) = y^1(x) & \text{in} \quad \Omega, \end{cases}$$

where 1_ω stands for the characteristic function of the subset ω. The energy dissipation law is then

$$\frac{dE(t)}{dt} = -\int_\omega |y_t|^2 dx.$$

Question: Do they exist $C > 0$ and $\gamma > 0$ such that

$$E(t) \leq Ce^{-\gamma t} E(0), \quad \forall t \geq 0,$$

for all solution of the dissipative system?

This is equivalent to an **observability property**: There exists $C > 0$ and $T > 0$ such that

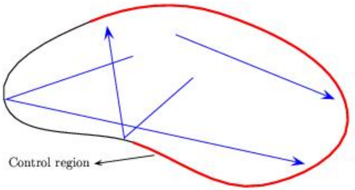
$$E(0) \leq C \int_0^T \int_\omega |y_t|^2 dxdt.$$

In other words, the exponential decay property is equivalent to showing that the dissipated energy within an interval $[0, T]$ contains a fraction of the initial energy, uniformly for all solutions. This estimate, together with the energy dissipation law, shows that

$$E(T) \leq \sigma E(0)$$

with $0 < \sigma < 1$. Accordingly the semigroup map $S(T)$ is a strict contraction. By the semigroup property one deduces immediately the exponential decay rate.

The observability inequality and, accordingly, the exponential decay property **holds if and only if the support of the dissipative mechanism, Γ_0 or ω, satisfies the so called the Geometric Control Condition (GCC)** (Ralston, Rauch-Taylor, Bardos-Lebeau-Rauch,...)



Rays propagating inside the domain Ω following straight lines that are reflected on the boundary according to the laws of Geometric Optics. The control region is the red subset of the boundary. The GCC is satisfied in this case. The proof requires tools from Microlocal Analysis.

The optimal shape design problem.

Given a subdomain ω (or Γ_0) for which the stabilization problem holds, it is natural to address the problem of optimizing the profile of the damping potential $a = a(x)$ to enhance the exponential decay rate. Consider

$$\begin{cases} y_{tt} - \Delta y = -a(x) y_t 1_\omega & \text{in} \quad Q = \Omega \times (0, \infty) \\ y = 0 & \text{on} \quad \Sigma = \Gamma \times (0, \infty) \\ y(x,0) = y^0(x), y_t(x,0) = y^1(x) & \text{in} \quad \Omega. \end{cases}$$

Then, for any $a > 0$ the exponential decay property holds:

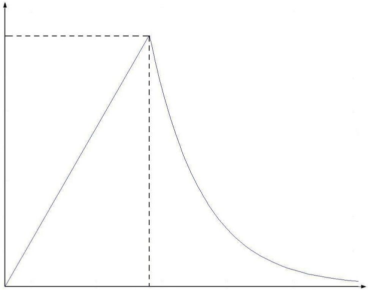
$$E(t) \leq C e^{-\gamma_a t} E(0), \quad \forall t \geq 0.$$

Obviously, the decay rate γ_a depends on the damping potential a.

It is therefore natural:

- To analyze the nature of the mapping $a \to \gamma_a$.
- One could also analyze the dependence of the decay rate on the geometry of the subdomain ω (γ_a depends also on ω).

Against the very first intuition this map is not monotonic with respect to the size of the damping. A $1-d$ spectral computation for constant coefficients yields:



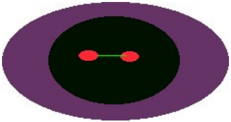
Some known results:

- $1-d$: The exponential decay rate coincides with the spectral abscissa within the class of BV damping potentials. For large eigenvalues $Re(\lambda) \sim -\int_\omega a(x)dx/2$ (S. Cox & E. Z., CPDE, 1993). Thus:

  $$\gamma_a \leq \int_\omega a(x)dx.$$

  Despite of the overdamping phenomenon, the following surprising result was proved (Castro-Cox, SICON, 2001): The decay rate may be made arbitrarily large by approximating singular potentials of the form $a(x) = 2/x$ for the space interval $\Omega = (0, 1)$. Note that is linked to the well known efficiency for the method of Perfectly Matching Layers (PML) for the computation of waves, as an alternative to transparent boundary conditions.

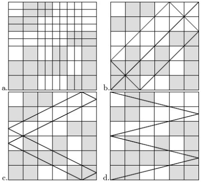
- In the multidimensional case the situation is even more complex. In this case it is not longer true that the spectral absicssa characterizes the exponential decay rate. There are actually two quantities that enter in such characterization (G. Lebeau, 1996):
- The spectral abscissa;
- The minimum asymptotic average (as $T \to \infty$) of the damping potential along rays of Geometric Optics. The later is in agreement with our intuition of waves traveling along rays of Geometric Optics.

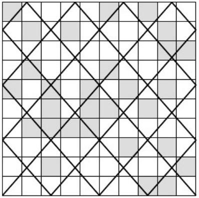


This is a typical situation in which the spectral abscissa does not suffice to capture the decay rate. The damping mechanism is active on the outer neighborhood of the exterior boundary. When the domain is the ellipsoid this produces the exponential decay. But, in the presence of the two holes, the exponential decay rate is lost, due to the existence of a trapped ray that never meets the damping region. In this case the decay rate is zero but the spectrum is not essentially affected if the holes are small enough. Thus the spectrum is unable to characterize the null decay rate.

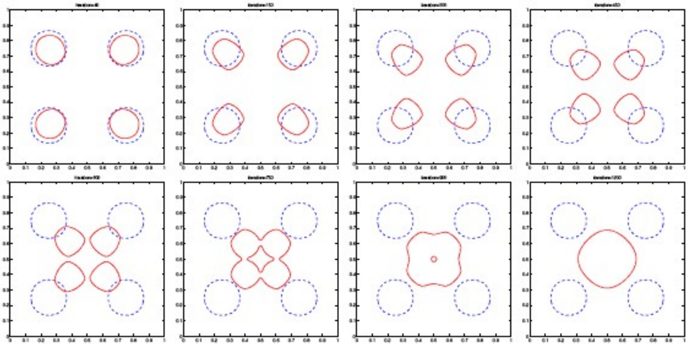
The optimal design of the damping potential with constraints (size, shape, etc.) is still widely open.

- Hébrard-Henrot, SCL, 2003. They show the complexity of the problem in the $1-d$ case for small amplitude damping potentials located on the union of a finite number of intervals.
- Hébrard-Humbert, 2003: Optimization of the shape of ω in a square domain in view of the geometric optics quantity entering in the characterization of the decay rate.
- Cox-Henrot, Ammari-Tucsnak, 2002: $1-d$ problems with damping terms located at a single point through a Dirac delta. Eigenvalues are complex valued, and they depend both on the amplitude of the damping and the diophantine properties of the point support.
- A. Münch, P. Pedregal, F. Periago 2005, ...: Young measures, relaxation, Level set methods.
- And many others...





Hébrard-Humbert, 2003



A. Munch, 2005.

The main difficulties are related to the fact that there is no variational principle characterizing the decay rate, and to the complex way in which the eigenvalues depend on the damping potentials, and the different way they do it for low/high/intermediate frequencies, for small/large amplitudes of the damping potentials, with respect to the shape of the support, ...

Futhermore, not always all authors deal with the same problem. For instance, the optimal damping for a given initial datum may differ significantly from the optimal damping when considering globally all possible solutions...

This is the case even for constant damping potentials k. The optimal damping for the ℓ-th eigenfunction is $k = 2\sqrt{\mu_\ell}$.

Open problem # 1.1: Characterize the optimal dampers for given initial data. How do they depend on their regularity? What about initial data with a finite number of Fourier components?

Open problem # 1.2: Given the subdomain ω, characterize the optimal damping potential for all finite energy solutions.

Open problem # 1.3: Given a total amount of possible damping, to characterize the optimal subdomain ω for its location.

Open problem # 1.4: Optimal dampers for the billiard. What is the subdomain that absorbs faster all rays? How this depends on the geometry of Ω? How it depends on the number of connected components of ω and on its size? What about variable coefficients/metrics?

Let us now report on some recent joint work with **Y. Privat** and **E. Trélat** that indicates what the expected answer could be.
Our analysis is concerned with the very closely related problems of *optimal placement of observers and controllers* for the conservative dynamics, in one space dimension.

$$\begin{cases} y_{tt} - y_{xx} = 0 & \text{in} \quad Q = (0, \pi) \times (0, T) \\ y = 0 & \text{for} \quad x = 0, \pi; \quad t \in (0, T \\ y(x, 0) = y^0(x), y_t(x, 0) = y^1(x) & \text{in} \quad (0, \pi). \end{cases}$$

The problem is then of variational nature!

We consider four different problems. All concern the search of the optimal subset ω with a given measure $|\omega| = L$, $0 < L < \pi$ so that:
P1.- For fixed finite energy initial data (y^0, y^1), the energy concentrated in ω better captures the total one.
P2.- Same question but uniformly on the whole energy space for (y^0, y^1).
P3.- For fixed finite energy initial data (y^0, y^1), the cost of controlling the system by acting on ω is minimized.
P4.- Same question but uniformly on the whole energy space for (y^0, y^1).

The following results are proved:

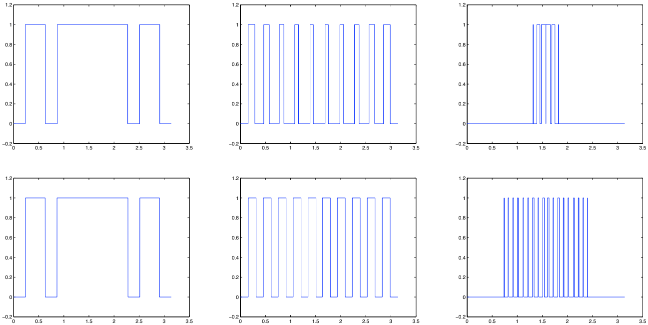
1.- Problems P1 & P3: For initial data that are analytic (exponential decay of Fourier coefficients), there is a unique minimizer with a finite number of connected components.

2.- Problem P1: The optimal set always exists but it can be a **Cantor set**.

3.- Problem P2: Relaxation occurs (Hebrart-Henrot): the optimum is achieved by a density function $\rho(x)$ so that $\int_0^\pi \rho(x)dx = L$ and not by a measurable set with bang-bang densities. In our work we actually prove the infima of both the relaxed and the classical problem coincide.

3.- Problem P3: There are (rough) initial data for which the optimal domain does not exist. There is a relaxation phenomenon so that

4.- Problem P4: Relaxation occurs.



Simulations performed using AMPL + IPOPT.

Sharp observability estimates for heat equations

THE CONTROL PROBLEM

Let $n \geq 1$ and $T > 0$, Ω be a simply connected, bounded domain of $\mathbb{R}^n$ with smooth boundary Γ, $Q = (0, T) \times \Omega$ and $\Sigma = (0, T) \times \Gamma$:

$$\begin{cases} u_t - \Delta u = f 1_\omega & \text{in} \quad Q \\ u = 0 & \text{on} \quad \Sigma \\ u(x,0) = u^0(x) & \text{in} \quad \Omega. \end{cases} \tag{22}$$

1_ω denotes the characteristic function of the subset ω of Ω where the control is active.

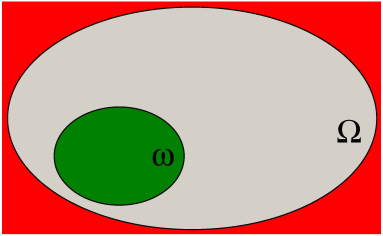
We assume that $u^0 \in L^2(\Omega)$ and $f \in L^2(Q)$ so that (22) admits an unique solution

$$u \in C\left([0, T]\,; L^2(\Omega)\right) \cap L^2\left(0, T; H_0^1(\Omega)\right).$$

$u = u(x,t) = \textit{solution} = \textit{state}, \; f = f(x,t) = \textit{control}$

Well known result (Fursikov-Imanuvilov, Lebeau-Robbiano,...) :
The system is null-controllable in any time T and from any open non-empty subset ω of Ω.
In other words, for all $u_0 \in L^2(\Omega)$ there exists a control $f \in L^2(\omega \times (0, T))$ such that the corresponding solution satisfies

$$u(T) \equiv 0.$$



The control of minimal L^2-norm can be found by minimizing

$$J_0(\varphi^0) = \frac{1}{2}\int_0^T \int_\omega \varphi^2 dxdt + \int_\Omega \varphi(0)u^0 dx \tag{23}$$

over the space of solutions of the adjoint system:

$$\begin{cases} -\varphi_t - \Delta\varphi = 0 & \text{in } Q \\ \varphi = 0 & \text{on } \Sigma \\ \varphi(T,x) = \varphi^0(x) & \text{in } \Omega. \end{cases} \tag{24}$$

Obviously, the functional is continuous and convex from $L^2(\Omega)$ to $\mathbb{R}$ and coercive because of the observability estimate:

$$\| \varphi(0) \|^2_{L^2(\Omega)} \leq C\int_0^T \int_\omega \varphi^2 dxdt, \quad \forall \varphi^0 \in L^2(\Omega). \tag{25}$$

This estimate,

$$\| \varphi(0) \|_{L^2(\Omega)}^2 \leq C \int_0^T \int_\omega \varphi^2 dxdt, \quad \forall \varphi^0 \in L^2(\Omega), \tag{26}$$

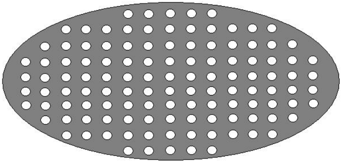
was proved by Fursikov and Imanuvilov (1996) using Carleman inequalities. In fact the same proof applies for equations with smooth (C^1) variable coefficients in the principal part and for heat equations with lower order potentials.
On has in fact

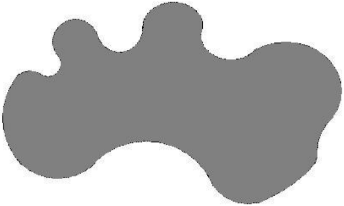
$$\int_0^T \int_\Omega e^{\frac{-A}{(T-t)}} \varphi^2 dxdt \leq C \int_0^T \int_\omega \varphi^2 dxdt.$$

Open problem # 2.1: Characterize the best constant A in this inequality:

$$A = A(\Omega, \omega).$$

The Carleman inequality approach allows establishing some upper bounds on A depending on the properties of the weight function. But this does not give a clear path towards the obtention of a sharp constant.





Lower bounds.

L. Miller (2003) , by inspection of the heat kernel, proved

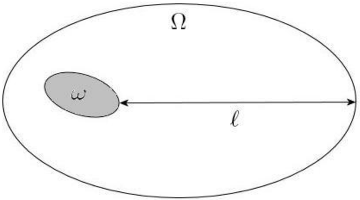
$$A > \ell^2/2$$

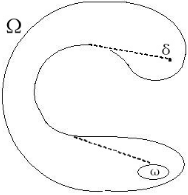
where ℓ is the length of the largest geodesic in $\Omega \setminus \omega$.
Recall that:

$$G(x,t) = (4\pi t)^{-n/2} \exp\left(\frac{-|x|^2}{4t}\right).$$

then, the following upper bound holds for the Green function in Ω:

$$G_\Omega(x,y,t) \leq Ct^{-n/2} \exp\left(\frac{-d^2(x,y)}{(4+\delta)t}\right).$$





Open problem # 2.2: To get sharp lower bounds. Can the lower bound $A > \ell^2/2$ be improved?

Note it is hard to guess any better lower bound. This would amount to find solutions of the heat equation exhibiting higher concentration effects than the Gaussian heat kernel itself.

Upper bounds.

Several works have also been devoted to get upper bounds on the best constant A using Carleman inequalities, Kannai's tranform and the control of waves under the so-called Geometric Control Condition (GCC) (Miller), one-dimensional tools from non-harmonic Fourier series, moment problems and number theory (Fatorinni-Russell, Seidman; Tucsnak and Tenenbaum,...).
But, as far as we know, until recently the only sharp result was the one by Fatorinni & Russell (1971) showing that $A = \ell^2/2$ in one space dimension.

More recently, in a joint work with S. Ervedoza we have shown that, whenever the GCC is fulfilled, for time T, then we have the upper bound:

$$A \leq T^2/8.$$

Note that for a ball Ω, with control on a neighborhood of the boundary,

$$T = 2\ell.$$

We thus get the sharp upper bound in this case:

$$A \leq \ell^2/2.$$

We use an **inverse Kannai transform.**

The **Kannai transform** allows transferring the results we have obtained for the wave equation to other models and in particular to the heat equation (Y. Kannai, 1977; K. D. Phung, 2001; L. Miller, 2004)

$$e^{t\Delta}\varphi = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{+\infty} e^{-s^2/4t} W(s) ds$$

where $W(x, s)$ solves the corresponding wave equation with data $(\varphi, 0)$.

$$W_{ss} - \Delta W = 0 \quad + \quad K_t - K_{ss} = 0 \quad \rightarrow \quad U_t - \Delta U = 0,$$

$$W_{ss} - \Delta W = 0 \quad + \quad iK_t - K_{ss} = 0 \quad \rightarrow \quad iU_t - \Delta U = 0.$$

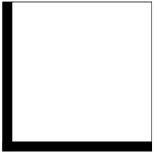
This can be actually applied in a more general abstract context $(U_t + AU = 0)$ but not when the equation has time-dependent coefficients.

Our proof is based on an inverse Kannais transform that, to the best of our knowledge, was unknown until now:

$$W(s) = \int_{\mathbf{R}_+} \frac{1}{(4\pi t)^{1/2}} \sin\left(\frac{sS}{2t}\right) \exp\left(\frac{s^2 - S^2}{4t}\right) U(t)\, dt.$$

Note however, that, even under the GCC there are no sharp upper bounds for other domains. For instance for **the square with observation on two consecutive sides** we have:

$$\frac{1}{2} \leq A \leq 1.$$



Open problem # 2.3: Get sharp upper bounds for other domains fulfilling the GCC.

Open problem # 2.4: Get some upper bounds for domains that do not fulilll GCC.

Open problem # 2.5: Possible connections with well known results on decay rates for damped wave equations in which both microlocal quantities and spectral ones enter, that only coincide in $1 - d$ (see Section #1)???

Robust control of linear finite-dimensional systems

Partially dissipative linear hyperbolic systems

$$\frac{\partial w}{\partial t}+\sum_{j=1}^{m} A_j \frac{\partial w}{\partial x_j} = -Bw \ , \ x\in\mathbb{R}^m \ , \ w\in\mathbb{R}^n \qquad (27)$$

$$\begin{array}{c|c} \begin{array}{c} A_1, \ldots, A_m \\ \text{symmetric} \end{array} & B = \begin{array}{cc} \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix} & \begin{array}{l} \updownarrow n_1 \\ \updownarrow n_2 \end{array} \end{array} \qquad \begin{array}{c} X^t DX > 0 \\ \forall X\in\mathbb{R}^{n_2}-\{0\} \end{array} \end{array}$$

Goal: Understand the asymptotic behavior as $t\to\infty$.

Apply Fourier transform:

$$\frac{\partial \hat{w}}{\partial t} = (-B - iA(\xi))\hat{w} \qquad \text{where} \qquad A(\xi) := \sum_{j=1}^{m} \xi_j A_j$$

Lack coercivity :

$\langle [B+iA(\xi)]X, X\rangle = \langle BX, X\rangle = \langle DX_2, X_2\rangle \not\geqslant c|X|^2$

is compatible with the decay depending on ξ:

$$\exp[(-B-iA(\xi))t] \leqslant Ce^{-\lambda(\xi)t}$$

PARTIALLY DISSIPATIVE LINEAR HYPERBOLIC SYSTEM

$\equiv$

m-PARAMETER (ξ) FAMILY OF FINITE-DIMENSIONAL

A quantitative measure of the decay rate as a function of ξ:

$$\begin{array}{c} A_1, ..., A_m \\ \text{symmetric} \end{array} \quad \Bigg| \quad B = \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix} \quad \begin{array}{l} \updownarrow n_1 \\ \updownarrow n_2 \end{array} \quad \Bigg| \quad A(\xi) := \sum_{j=1}^{m} \xi_j A_j$$

$\xi = \rho\omega \in \mathbb{R}^m \qquad \rho > 0 \qquad \omega \in S^{m-1} \qquad (m_k) \uparrow$ well chosen

$$N_{*,\epsilon}(\omega) := \min\{\sum_{k=0}^{n-1} \epsilon^{m_k} |BA(\omega)^k x|^2; x \in S^{n-1}\}.$$

Theorem

(K. Beauchard and E. Z.)
$\exists \epsilon_* > 0, c > 0$ *such that* $\forall \epsilon \in (0, \epsilon_*)$,

$$\exp[(-B - i\rho A(\omega))t] \leqslant 2e^{-cN_{*,\epsilon}(\xi)\min\{1,\rho^2\}t}.$$

Remark : **(SK)= (Shizuta-Kawashima)** $\Leftrightarrow$ **Kalman rank condition for** (A, B) $\Leftrightarrow N_{*,\epsilon}(\omega) \geqslant N_{*,\epsilon} > 0, \forall \omega \in S^{m-1}$.
In general, $N_{*,\epsilon}(\omega)$ may vanish for some values of $\omega \in S^{m-1}$, in which case the decomposition of solutions and its asymptotic form is more complex.

The set of degeneracy :
$\mathcal{D}(B + iA(\xi)) = \{\xi \in \mathbb{R}^m; \mathrm{rank}[B|BA(\xi)|...|BA(\xi)^{n-1}] < n\}$
is an algebraic submanifold

- either $|\mathcal{D}| = 0 \Leftrightarrow N_{*,\epsilon} > 0$ a.e. $\Rightarrow$ strong L^2 stability;
  or
- $\mathcal{D} = \mathbb{R}^m$: $\exists$ non dissipated solutions

Open problem # 3.1: Characterize and classify, in terms of (A, B), the possible sets of degeneracy $\mathcal{D}$.
Open problem # 3.2: Characterize and classify, in terms of (A, B), the possible degenerate behaviors of $N_{*,\epsilon}(\omega)$ as $\omega \to \mathcal{D}$.
Open problem # 3.3: Classify the possible asymptotic behaviors of partially dissipative hyperbolic systems as $t \to \infty$.
Open problem # 3.4: Describe the controllability properties of m-parameter families of finite-dimensional systems:

$$x'(t) + iA(\xi)x(t) = Bu(t) \qquad \text{where} \qquad A(\xi) := \sum_{j=1}^{m} \xi_j A_j.$$

An example:

Theorem

(K. Beauchard & E. Z.)

When $n_1 = 1$, $\mathcal{D}$ is a vector subspace of $\mathbb{R}^m$ and

$$N_{*,\epsilon}(\omega) \geqslant c \min\{1, dist(\omega, \mathcal{D})^2\}, \forall \omega \in S^{m-1}.$$

Example: $n = m = 2$; $\mathcal{D} = \{(\xi_1, \xi_2) : a^1_{21}\xi_1 + a^2_{21}\xi_2 = 0\}$.

Control of Kolmogorov's equation

Null control of the Kolmogorov equation:

$$\frac{\partial f}{\partial t}+v\frac{\partial f}{\partial x}-\frac{\partial^2 f}{\partial v^2}=u(t,x,v)1_\omega(x,v),(x,v)\in \mathbf{R}_x\times\mathbf{R}_v, t\in(0,+\infty). \tag{28}$$

In a recent work with K. Beauchard, we consider the particular case where where $\omega = \mathbf{R}_x \times \Big[\mathbf{R}_v - [a,b]\Big]$.

Equivalently, one may address the following observability inequality for the adjoint system:

$$\begin{cases} \frac{\partial g}{\partial t} - v\frac{\partial g}{\partial x} - \frac{\partial^2 g}{\partial v^2} = 0, (x,v) \in \mathbf{R}_x \times \mathbf{R}_v, t \in (0,T), \\ g(0,x,v) = g_0(x,v), (x,v) \in \mathbf{R}_x \times \mathbf{R}_v. \end{cases} \tag{29}$$

$$\int_{\mathbf{R}_x \times \mathbf{R}_v} |g(T,x,v)|^2 dxdv \leqslant C \int_0^T \int_\omega |g(t,x,v)|^2 dxdvdt.$$

Theorem

(K. Beauchard and E. Z.)

In the particular case where $\omega = \mathbf{R}_x \times \Big[\mathbf{R}_v - [a,b]\Big]$ *the observability inequality holds for the adjoint system and the Kolmogorov equation is null controllable.*

Ideas of the proof:

- Fourier transform in v:

$$\begin{cases} \frac{\partial \hat{f}}{\partial t}(t,\xi,v) + i\xi v \hat{f}(t,\xi,v) - \frac{\partial^2 \hat{f}}{\partial v^2}(t,\xi,v) = \hat{u}(t,\xi,v) 1_{\mathbb{R}-[a,b]}(v), \\ \hat{f}(0,\xi,v) = \hat{f}_0(\xi,v). \end{cases} \tag{30}$$

- Decay:

$$\left| \hat{f}(t,\xi,.) \right|_{L^2(\mathbb{R})} \leqslant \left| \hat{f}_0(\xi,.) \right|_{L^2(\mathbb{R})} e^{-\xi^2 t^3/12}, \forall \xi \in \mathbb{R}, \forall t \in \mathbb{R}_+. \tag{31}$$

- Control depending on the parameter ξ with cost

$$e^{C(T)\max\{1,\sqrt{|\xi|}\}}.$$

The exponentially large cost of control for high frequencies is compensated by the exponential (and stronger) decay rate.

Open problem # 4.1: Similar results hold for other geometries of control sets?

Open problem # 4.2: What about more general classes of hypoelliptic equations?

Open problem # 4.3: May Carleman inequalities be applied directly on the Kolmogorov system without using Fourier transform?

Open problem # 4.4: How are related the notions of hypoellipticity and hypocoercivity with the property of null controllability (connections with Open Problems #2.X on the heat kernel).

References

E. ZUAZUA, "Controllability and Observability of Partial Differential Equations: Some results and open problems", in *Handbook of Differential Equations: Evolutionary Equations, vol. 3*, C. M. Dafermos and E. Feireisl eds., Elsevier Science, 2006, pp. 527-621.